The Irreducible Representations Of Rational Cherednik Algebras Associated To The Symmetric Groups S_2 And S_3 In Positive Characteristic

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For my family.

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Abstract

Let $\Bbbk$ be an algebraically closed field of positive characteristic p. Given parameters $t, c \in \Bbbk$ we form the rational Cherednik algebra $H_{t,c}(S_n, \mathfrak{h})$ for $n = 2, 3$ and consider its representation theory over $\Bbbk$. For each irreducible representation τ of S_n there is a Verma module $M_{t,c}(\tau)$ which is a module for $H_{t,c}(S_n, \mathfrak{h})$ and this module has a unique maximal proper graded submodule $J_{t,c}(\tau)$. The quotient $M_{t,c}(\tau)/J_{t,c}(\tau)$ is a graded, irreducible representation $L_{t,c}(\tau)$ of $H_{t,c}(S_n, \mathfrak{h})$ in category $\mathcal{O}$ and all simple objects in $\mathcal{O}$ are of this form. Our goal is to describe the representations $L_{t,c}(\tau)$ for all possible values of p, t, c, and τ, by calculating their characters, Hilbert polynomials, and specifying the generators of $J_{t,c}(\tau)$. We achieve this goal, filling gaps in the literature and describing these modules completely and explicitly, in all cases except one where we provide a conjecture.

Introduction

Rational Cherednik algebras were introduced in 2002 by Pavel Etingof and Victor Ginzburg [EtGi02], building on earlier work of Ivan Cherednik [Ch92, Ch95]. Since then, the representation theory of these algebras has been studied extensively. The vast majority of work on the representation theory of rational Cherednik algebras takes place over fields of characteristic 0 [DJO94, BEG03a, BEG03b, ChEt03, GGOR03, Go03, Du04, Gr08]. Although much less is known in positive characteristic, several papers have been published [La05, BFG06, BeMa13, BaCh13a, BaCh13b, DeSa14, DeSu16, CaKa21].

Rational Cherednik algebras are constructed from reflection groups, such as the symmetric groups. In our work, we study the representation theory of rational Cherednik algebras constructed from the symmetric groups S_2 and S_3 over an algebraically closed field $\Bbbk$ of positive characteristic p. Given two parameters $t, c \in \Bbbk$, we denote by $H_{t,c}(S_n, \mathfrak{h})$ the rational Cherednik algebra associated to the symmetric group S_n. The representation theory of $H_{t,c}(S_n, \mathfrak{h})$ falls into distinct situations depending on its parameters.

The first dichotomy is that $t = 0$ behaves differently than $t = 1$. Secondly, the cases $p \leq n$ and $p > n$ have distinct behaviour, related to the representation theory of S_n in positive characteristic. Finally, the parameter c has one behaviour for generic values, but can have special behaviour at particular values. In characteristic $p > n$ the generic values are $c \neq 0$ when $t = 0$ and $c \notin \mathbb{F}_p$ when $t = 1$; conversely, when $t = 0$ the value $c = 0$ is a special case, and when $t = 1$ the values of $c \in \{0, 1, 2, \ldots, p-1\}$ are special.

We perform the customary task of describing irreducible modules for our algebra in some fixed category. We study the category $\mathcal{O}$ as defined in [BaCh13a], which is analogous to categories $\mathcal{O}$ found in other contexts, but the definition is somewhat different to make it suitable for nonzero characteristic.

As rational Cherednik algebras are constructed from reflection groups, we can construct modules for rational Cherednik algebras from representations of reflection groups. For each irreducible representation τ of S_n we can construct a Verma module, denoted $M_{t,c}(\tau)$, which is an induced graded module for $H_{t,c}(S_n, \mathfrak{h})$. These Verma modules are the standard objects in our work and, as usual, they have unique irreducible quotients, denoted $L_{t,c}(\tau)$, which belong to the category $\mathcal{O}$. Therefore in order to understand Verma modules (and consequently their irreducible quotients) it is important to understand the representation theory of the symmetric group.

For instance, the irreducible representations of S_n vary depending on the characteristic p of $\Bbbk$. In characteristic 0 and characteristic $p > 2$, the symmetric group S_2 has two irreducible representations up to isomorphism, `triv` and `sign`, while in characteristic $p = 2$ there is only one, `triv`. Similarly, in characteristic 0 and characteristic $p > 3$, S_3 has three irreducible representations up to isomorphism (`triv`, `sign` and `stand`) but in characteristic $p = 3$ there are only two, which are `triv` and `sign`, whereas in characteristic $p = 2$ there are only two, which are `triv` and `stand`. In positive characteristic, $M_{t,c}(\tau)$ has a unique maximal proper graded submodule denoted $J_{t,c}(\tau)$ with $L_{t,c}(\tau) = M_{t,c}(\tau)/J_{t,c}(\tau)$ and the $L_{t,c}(\tau)$ are all finite-dimensional and graded. The behaviour of Verma modules (and consequently their irreducible quotients) varies as we vary the parameters p, t, c and τ.

The aim of this thesis is to describe the irreducible modules $L_{t,c}(\tau)$ for $H_{t,c}(S_2, \mathfrak{h})$ and $H_{t,c}(S_3, \mathfrak{h})$ for all possible values of p, t, c and τ. We do this by giving their characters, Hilbert polynomials, and the generators of $J_{t,c}(\tau)$.

A crucial concept in our work is that of singular vectors, which are elements of a Verma module (or its quotients) that generate proper graded submodules. In order to obtain the irreducible quotient of a Verma module, we must take a quotient by the maximal proper graded submodule $J_{t,c}(\tau)$. Therefore it is important to understand the singular vectors, as they are the generators of $J_{t,c}(\tau)$. Some authors, such as [DeSa14], provide the character and Hilbert polynomials of irreducible modules but do not describe the singular vectors. Other authors, such as [DeSu16] and [CaKa21], provide the singular vectors and Hilbert polynomials, but do not provide characters. Our goal is to calculate the irreducible modules and provide a complete description of their characters, Hilbert polynomials, and singular vectors.

Our primary method utilises the grading on Verma modules to search for singular vectors systematically degree by degree. Some vectors are only singular modulo a quotient by other singular vectors of lower degree, therefore when searching for singular vectors we must begin in low degree and work up. We often use properties of the Casimir operator to further reduce the space in which we need to look. The Casimir operator acts by a fixed value on each irreducible S_n representation, such as those spanned by singular vectors. This, along with a graded property of the Casimir operator, allows us to restrict in which degrees singular vectors can occur, and even what form they can take. Where possible, we use a particular basis of Verma modules that simplifies our computations. We provide a basis of the Verma module $M_{t,c}(\tau)$ for $H_{t,c}(S_3, \mathfrak{h})$ which is compatible with the decomposition of each graded piece of the Verma module into irreducible S_3 representations. This basis is also primarily composed of symmetric polynomials which behave well in calculations.

The character of an irreducible module records how each graded piece decomposes into finite-dimensional representations of S_n, while the Hilbert polynomial records the dimension of each graded piece as a vector space. More precisely, the coefficients of the character represent congruence classes in the Grothendieck group of the category of finite dimensional representations of S_n. In characterisic $p > n$, if c is not generic then $c \in \{0, 1, \ldots, p-1\}$. We sometimes consider this value as an integer $c \in \mathbb{Z}$, such as when stating inequalities or when c appears in exponents, and other times it is an element $c \in \mathbb{F}_p \subseteq \Bbbk$, such as when it appears in the coefficients of a singular vector.

The following is a summary of the results in this thesis.

The part of Theorem 1 concerning characteristic $p \geq 3$ is partially proved in the work of [La05], as discussed in Chapter 5, but we consider all characteristics and additionally provide all characters.

The part of Theorem 2 concerning the case of $\tau =$ `triv` is partially proved in the work of [CaKa21] which provides the Hilbert polynomials and singular vectors. We prove the case of $\tau =$ `stand` and additionally calculate characters in all cases.

The part of Theorem 3 concerning the case of $\tau =$ `triv` is partially proved in the work of [DeSu16] which provides the Hilbert polynomials and singular vectors. We also consider the case of $\tau =$ `sign` and calculate characters in all cases.

The part of Theorem 4 concerning Hilbert series and characters is proved in the work of [DeSa14]. We contribute the singular vectors in all cases and demonstrate a novel approach to the proof using a particular basis.

The part of Theorem 5 concerning the case of $\tau =$ `triv` is partially proved by [Li14]. Our contribution is providing the characters, and also solving the remaining unknown case of $\tau =$ `triv` where $p/2 < c < 2p/3$. We prove new results in the case of $\tau =$ `stand` at special values of c which is a case that had not previously been considered.

Theorem 1. The characters and Hilbert polynomials of the irreducible representation $L_{t,c}(\tau)$ of the rational Cherednik algebra $H_{t,c}(S_2, \mathfrak{h})$, for any p, t, c and τ, are given by the following tables.

Characters:

$\boldsymbol{p=2}$	$\tau = \texttt{triv}$
$t=0$, all c	$[\texttt{triv}]$
$t=1$, all c	$[\texttt{triv}] + [\texttt{triv}]z$

$\boldsymbol{p>2}$	$\tau = \texttt{triv}$
$t=0,\ c=0$	$[\texttt{triv}]$
$t=0,\ c\neq 0$	$[\texttt{triv}] + [\texttt{sign}]z$
$t=1,\ c\notin \mathbb{F}_p$	$[\texttt{triv}]\dfrac{1-z^{2p}}{1-z^2} + [\texttt{sign}]\dfrac{z(1-z^{2p})}{1-z^2}$
$t=1$, $0\le c<p/2$	$[\texttt{triv}]\dfrac{1-z^{2c+p+1}}{1-z^2} + [\texttt{sign}]\dfrac{z(1-z^{2c+p-1})}{1-z^2}$
$t=1$, $p/2<c<p$	$[\texttt{triv}]\dfrac{1-z^{2c-p+1}}{1-z^2} + [\texttt{sign}]\dfrac{z(1-z^{2c-p-1})}{1-z^2}$

$\boldsymbol{p>2}$	$\tau = \texttt{sign}$
$t=0,\ c=0$	$[\texttt{sign}]$
$t=0,\ c\neq 0$	$[\texttt{sign}] + [\texttt{triv}]z$
$t=1,\ c\notin \mathbb{F}_p$	$[\texttt{sign}]\dfrac{1-z^{2p}}{1-z^2} + [\texttt{triv}]\dfrac{z(1-z^{2p})}{1-z^2}$
$t=1$, $0\le c<p/2$	$[\texttt{sign}]\dfrac{1-z^{-2c+p+1}}{1-z^2} + [\texttt{triv}]\dfrac{z(1-z^{-2c+p-1})}{1-z^2}$
$t=1$, $p/2<c<p$	$[\texttt{sign}]\dfrac{1-z^{-2c+3p+1}}{1-z^2} + [\texttt{triv}]\dfrac{z(1-z^{-2c+3p-1})}{1-z^2}$

Hilbert polynomials:

$\boldsymbol{p=2}$	$\tau = \texttt{triv}$
$t=0$, all c	1
$t=1$, all c	$1+z$

$p > 2$	$\tau = \texttt{triv}$
$t = 0,\ c = 0$	1
$t = 0,\ c \neq 0$	$1 + z$
$t = 1,\ c \notin \mathbb{F}_p$	$\frac{1 - z^{2p}}{1 - z}$
$t = 1,$ $0 \leq c < p/2$	$\frac{1 - z^{2c+p}}{1 - z}$
$t = 1,$ $p/2 < c < p$	$\frac{1 - z^{2c-p}}{1 - z}$

$p > 2$	$\tau = \texttt{sign}$
$t = 0,\ c = 0$	1
$t = 0,\ c \neq 0$	$1 + z$
$t = 1,\ c \notin \mathbb{F}_p$	$\frac{1 - z^{2p}}{1 - z}$
$t = 1,$ $0 \leq c < p/2$	$\frac{1 - z^{-2c+p}}{1 - z}$
$t = 1,$ $p/2 < c < p$	$\frac{1 - z^{-2c+3p}}{1 - z}$

In all cases, we also calculate the singular vectors.

Proof. This is Theorem 6.0.2 in Chapter 6. The proof for $t = 0$ is found in Section 6.1 and the proof for $t = 1$ is found in Section 6.2. □

Theorem 2. The characters and Hilbert polynomials of the irreducible representation $L_{t,c}(\tau)$ of the rational Cherednik algebra $H_{t,c}(S_3, \mathfrak{h})$ over an algebraically closed field of characteristic 2, for any t, c and τ, are given by the following tables.

Characters:

$p = 2$	$\tau = \texttt{triv}$
$t = 0,\ c = 0$	$[\texttt{triv}]$
$t = 0,\ c \neq 0$	$[\texttt{triv}] + [\texttt{stand}](z + z^2) + [\texttt{triv}]z^3$
$t = 1,\ c \notin \mathbb{F}_2$	$\big([\texttt{triv}] + [\texttt{stand}]z + [\texttt{triv}]z^2\big)\big([\texttt{triv}] + [\texttt{stand}]z^2 + [\texttt{stand}]z^4 + [\texttt{triv}]z^6\big)$
$t = 1,\ c = 0$	$[\texttt{triv}] + [\texttt{stand}]z + [\texttt{triv}]z^2$
$t = 1,\ c = 1$	$[\texttt{triv}]$

$p = 2$	$\tau = \texttt{stand}$
$t = 0,\ c = 0$	$[\texttt{stand}]$
$t = 0,\ c \neq 0$	$[\texttt{stand}] + ([\texttt{triv}] + [\texttt{sign}])z + [\texttt{stand}]z^2$
$t = 1,\ c \notin \mathbb{F}_2$	$[\texttt{stand}](1 + z + z^2 + 2z^3 + z^4 + z^5 + z^6) + 2[\texttt{triv}](z + z^2 + z^4 + z^5)$
$t = 1,\ c = 0$	$[\texttt{stand}] + \big([\texttt{stand}] + 2[\texttt{triv}]\big)z + [\texttt{stand}]z^2$
$t = 1,\ c = 1$	$[\texttt{stand}](1 + z + z^2 + 2z^3 + z^4 + z^5 + z^6) + [\texttt{triv}](z + 2z^2 + 2z^4 + z^5)$

Hilbert polynomials:

$\boldsymbol{p=2}$	$\tau = \texttt{triv}$	$\tau = \texttt{stand}$
$t=0,\ c=0$	1	2
$t=0,\ c\neq 0$	$1+2z+2z^2+z^3$ *[CaKa21], Thm 2.11*	$2+2z+2z^2$
$t=1,\ c\notin \mathbb{F}_2$	$\dfrac{(1-z^4)(1-z^6)}{(1-z)^2}$ *[CaKa21], Thm 3.17*	$\dfrac{2(1-z^2)(1-z^6)}{(1-z)^2}$
$t=1,\ c=0$	$(1+z)^2$	$2+4z+2z^2$
$t=1,\ c=1$	1 *[Li14], Thm. 3.2*	$\dfrac{2-z-z^3-z^5-z^7+2z^8}{(1-z)^2}$

In all cases, the singular vectors are known.

Proof. This is Theorem 8.0.1 in Chapter 8. The irreducible representation $L_{t,c}(\texttt{triv})$ is described in the following lemmas and propositions:

- for $t=0,\ c=0$ in Proposition 2.6.11 or Proposition 4.1.4;
- for $t=0,\ c\neq 0$ in Lemma 8.1.2;
- for $t=1,\ c\neq 0,1$ in Lemma 8.2.2;
- for $t=1,\ c=0$ in Lemma 2.6.13;
- for $t=1,\ c=1$ in Proposition 4.1.4.

The irreducible representation $L_{t,c}(\texttt{stand})$ is described in the following lemmas and propositions:

- for $t=0,\ c=0$ in Proposition 2.6.11;
- for $t=0, c\neq 0$ in Lemma 8.3.3;
- for $t=1,\ c\neq 0,1$ in Lemma 8.4.7 and Lemma 8.4.12;
- for $t=1,\ c=0$ in Lemma 2.6.13;
- for $t=1,\ c=1$ in Lemma 8.5.1.

□

Theorem 3. The characters and Hilbert polynomials of the irreducible representation $L_{t,c}(\tau)$ of the rational Cherednik algebra $H_{t,c}(S_3, \mathfrak{h})$ over an algebraically closed field of characteristic 3, for any t, c and τ, are given by the following tables.

$\boldsymbol{p=3}$	$\tau = \mathtt{triv}$
$t=0$, all c	$[\mathtt{triv}]$
$t=1$, all c	$[\mathtt{triv}](1+z+2z^2+z^3+z^4)+[\mathtt{sign}](z+z^2+z^3)$

$\boldsymbol{p=3}$	$\tau = \mathtt{sign}$
$t=0$, all c	$[\mathtt{sign}]$
$t=1$, all c	$[\mathtt{sign}](1+z+2z^2+z^3+z^4)+[\mathtt{triv}](z+z^2+z^3)$

$\boldsymbol{p=3}$	$\tau = \mathtt{triv}$	$\tau = \mathtt{sign}$
$t=0$, all c	1	1
$t=1$, all c	$\left(\frac{1-z^3}{1-z}\right)^2$ *[DeSu16], Thm 4.1*	$\left(\frac{1-z^3}{1-z}\right)^2$

In all cases, we calculate the singular vectors.

Proof. This is Theorem 9.0.1 in Chapter 9. The irreducible representation $L_{t,c}(\mathtt{triv})$ is described via its singular vectors, character and Hilbert polynomial in the following Lemmas:

- for $t=0$ and any c, in Lemma 9.1.1;
- for $t=1$ and any c, in Lemma 9.2.1.

The analogous descriptions of the irreducible representation $L_{t,c}(\mathtt{sign})$ can be deduced from the description of $L_{t,-c}(\mathtt{triv})$ and Corollary 2.8.3. Note that the Hilbert polynomials of $L_{1,c}(\mathtt{triv})$ for generic c are also known from Theorem 5.4.1 ([DeSu16], Theorem 4.1). □

Theorem 4. The characters and the Hilbert polynomials of the irreducible representation $L_{t,c}(\tau)$ of the rational Cherednik algebra $H_{t,c}(S_3, \mathfrak{h})$ over an algebraically closed field of characteristic $p > 3$, for generic c, $t = 0, 1$, and any τ, are given by the following tables.

Characters:

$\boldsymbol{p > 3}$	$t = 0$, $c \neq 0$ generic	$t = 1$, $c \notin \mathbb{F}_p$
$\tau = \texttt{triv}$	$[\texttt{triv}] + [\texttt{stand}](z + z^2) + [\texttt{sign}]z^3$ *[DeSa14], Prop 4.1*	$\chi_{S(\mathfrak{h}^*)}(z) \cdot (1 - z^{2p})(1 - z^{3p})$ *[DeSa14], Prop 4.2*
$\tau = \texttt{sign}$	$[\texttt{sign}] + [\texttt{stand}](z + z^2) + [\texttt{triv}]z^3$ *[DeSa14], Prop 4.1*	$\chi_{S(\mathfrak{h}^*)}(z) \cdot [\texttt{sign}](1 - z^{2p})(1 - z^{3p})$ *[DeSa14], Prop 4.2*
$\tau = \texttt{stand}$	$[\texttt{stand}] + ([\texttt{triv}] + [\texttt{sign}])z + [\texttt{stand}]z^3$ *[DeSa14], Prop 4.1*	$\chi_{S(\mathfrak{h}^*)}(z) \cdot [\texttt{stand}](1 - z^{p})(1 - z^{3p})$ *[DeSa14], Prop 4.2*

where

$$\chi_{S(\mathfrak{h}^*)}(z) = \frac{1}{(1 - z^2)(1 - z^3)} \left([\texttt{triv}] + [\texttt{stand}](z + z^2) + [\texttt{sign}]z^3\right).$$

Hilbert polynomials:

$\boldsymbol{p > 3}$	$t = 0$, $c \neq 0$ generic	$t = 1$, $c \notin \mathbb{F}_p$
$\tau = \texttt{triv}$	$1 + 2(z + z^2) + z^3$ *[DeSa14], Prop 4.1*	$\dfrac{(1 - z^{2p})(1 - z^{3p})}{(1 - z)^2}$ *[DeSa14], Prop 4.2*
$\tau = \texttt{sign}$	$1 + 2(z + z^2) + z^3$ *[DeSa14], Prop 4.1*	$\dfrac{(1 - z^{2p})(1 - z^{3p})}{(1 - z)^2}$ *[DeSa14], Prop 4.2*
$\tau = \texttt{stand}$	$2 + 2z + 2z^2$ *[DeSa14], Prop 4.1*	$\dfrac{2(1 - z^{p})(1 - z^{3p})}{(1 - z)^2}$ *[DeSa14], Prop 4.2*

The characters and Hilbert polynomials are already known from [DeSa14]. Our primary contribution is providing formulas for the singular vectors in all cases.

Proof. This is Theorem 11.0.1 in Chapter 11. The generic values of c ($c \neq 0$ for $t = 0$ and $c \notin \mathbb{F}_p$ for $t = 1$) are given in Proposition 4.1.3. For those values, the characters and Hilbert polynomials are given by [DeSa14] in Propositions 4.1 and 4.2 of their paper [DeSa14], as discussed in Proposition 5.3.1 and Proposition 5.3.2 of Section 5.3.

For $t = 0, 1$ and $\tau = \texttt{triv}, \texttt{sign}$, the Hilbert polynomial of $L_{t,c}(\tau)$ for generic c from Proposition 5.3.1 and Proposition 5.3.2 coincides with the Hilbert polynomials of $N_{t,c}(\tau)$ from

Examples 4.1.1 and 4.1.2, so we conclude that $L_{t,c}(\tau) = N_{t,c}(\tau)$. In this case all the singular vectors are known; for $t = 0$ they are $\sigma_i \otimes v$ for $i = 2, 3$ and $v \in \tau$, and for $t = 1$ they are $\sigma_i^p \otimes v$ for $i = 2, 3$ and $v \in \tau$.

For $t = 0, 1$ and $\tau = \texttt{stand}$, comparing the Hilbert polynomials of $L_{t,c}(\texttt{stand})$ for generic c from Proposition 5.3.1 and Proposition 5.3.2 with the Hilbert polynomials of of $N_{t,c}(\texttt{stand})$ from Examples 4.1.1 and 4.1.2 shows that $L_{t,c}(\tau)$ is a proper quotient of $N_{t,c}(\tau)$. For $t = 0$ the singular vectors are computed in Lemma 11.1.1, and alternatively are available in a different basis in Corollary 8.3.5. For $t = 1$ the singular vectors are computed in Lemma 11.2.8, and the Hilbert polynomial of the quotient of $N_{1,c}(\texttt{stand})$ by these singular vectors is computed in Lemma 11.2.19. Observing this polynomial is equal to the Hilbert polynomial of $L_{1,c}(\texttt{stand})$, we conclude this quotient is irreducible. Its character is then straightforward to compute. □

Theorem 5. The irreducible representations $L_{t,c}(\tau)$ of the rational Cherednik algebra $H_{t,c}(S_3, \mathfrak{h})$ over an algebraically closed field of characteristic $p > 3$, for special c, $t = 0, 1$, and any τ, are described by the following tables.

Characters:

$\boldsymbol{p > 3}$	$\tau = \texttt{triv}$
$t = 0,\ c = 0$	$[\texttt{triv}]$
$t = 1,\ c = 0$	$\chi_{S^{(p)}(\mathfrak{h}^*)}(z)$
$t = 1,$ $0 < c < p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{triv}] - [\texttt{stand}]z^{3c+p} + [\texttt{sign}]z^{2(3c+p)}\big)$
$t = 1,$ $p/3 < c < p/2$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{triv}] - [\texttt{stand}]z^{3c-p} + [\texttt{sign}]z^{2(3c-p)}\big)$
$t = 1,$ $p/2 < c < 2p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{triv}] - [\texttt{sign}]z^{6c-3p}\big)(1 - z^p)$
$t = 1,$ $2p/3 < c < p$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{triv}] - [\texttt{stand}]z^{3c-2p} + [\texttt{sign}]z^{2(3c-2p)}\big)$

$p > 3$	$\tau = \texttt{sign}$
$t = 0,\ c = 0$	$[\texttt{sign}]$
$t = 1,\ c = 0$	$\chi_{S^{(p)}(\mathfrak{h}^*)}(z) \cdot [\texttt{sign}]$
$t = 1,$ $0 < c < p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{sign}] - [\texttt{stand}]z^{p-3c} + [\texttt{triv}]z^{2(p-3c)}\big)$
$t = 1,$ $p/3 < c < p/2$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{sign}] - [\texttt{triv}]z^{-6c+3p}\big)(1 - z^p)$
$t = 1,$ $p/2 < c < 2p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot ([\texttt{sign}] - [\texttt{stand}]z^{2p-3c} + [\texttt{triv}]z^{2(2p-3c)})$
$t = 1,$ $2p/3 < c < p$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{sign}] - [\texttt{stand}]z^{4p-3c} + [\texttt{triv}]z^{2(4p-3c)}\big)$

$p > 3$	$\tau = \texttt{stand}$
$t = 0,\ c = 0$	$[\texttt{stand}]$
$t = 1,\ c = 0$	$\chi_{S^{(p)}(\mathfrak{h}^*)}(z) \cdot [\texttt{stand}]$
$t = 1,$ $0 < c < p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot ([\texttt{stand}] - [\texttt{triv}]z^{p-3c} - [\texttt{stand}]z^p - [\texttt{sign}]z^{p+3c} + 2[\texttt{sign}]z^{2p})$
$t = 1,$ $p/3 < c < p/2$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{stand}] - [\texttt{sign}]z^{-p+3c} - [\texttt{triv}]z^{3p-3c} - [\texttt{sign}]z^{p+3c} - [\texttt{triv}]z^{5p-3c} + [\texttt{stand}]z^{4p}\big)$
$t = 1,$ $p/2 < c < 2p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{stand}] - [\texttt{triv}]z^{-3c+2p} - [\texttt{sign}]z^{3c} - [\texttt{triv}]z^{-3c+4p} - [\texttt{sign}]z^{3c+2p} + [\texttt{stand}]z^{4p}\big)$
$t = 1,$ $2p/3 < c < p$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot ([\texttt{stand}] - [\texttt{sign}]z^{3c-2p} - [\texttt{stand}]z^p - [\texttt{triv}]z^{4p-3c} + 2[\texttt{triv}]z^{2p})$

with

$$\chi_{S(\mathfrak{h}^*)}(z) = \frac{1}{(1-z^2)(1-z^3)}\left([\texttt{triv}] + [\texttt{stand}](z+z^2) + [\texttt{sign}]z^3\right),$$
$$\chi_{S^{(p)}(\mathfrak{h}^*)}(z) = \chi_{S(\mathfrak{h}^*)}(z) \cdot (1 - [\texttt{stand}]z^p + [\texttt{sign}]z^{2p}).$$

Hilbert polynomials:

$\boldsymbol{p > 3}$	$\tau = \texttt{triv}$	$\tau = \texttt{sign}$
$t = 0,\ c = 0$	1	1
$t = 1$		
$c = 0$	$\left(\frac{1 - z^p}{1 - z}\right)^2$	$\left(\frac{1 - z^p}{1 - z}\right)^2$
$0 < c < p/3$	$\left(\frac{1 - z^{3c+p}}{1 - z}\right)^2$	$\left(\frac{1 - z^{p-3c}}{1 - z}\right)^2$
$p/3 < c < p/2$	$\left(\frac{1 - z^{3c-p}}{1 - z}\right)^2$	$\frac{(1 - z^{3p-6c})(1 - z^p)}{(1 - z)^2}$
$p/2 < c < 2p/3$	$\frac{(1 - z^{6c-3p})(1 - z^p)}{(1 - z)^2}$	$\left(\frac{1 - z^{2p-3c}}{1 - z}\right)^2$
$2p/3 < c < p$	$\left(\frac{1 - z^{3c-2p}}{1 - z}\right)^2$	$\left(\frac{1 - z^{4p-3c}}{1 - z}\right)^2$
	[Li14] (partial)	

$\boldsymbol{p > 3}$	$\tau = \texttt{stand}$
$t = 0,\ c = 0$	2
$t = 1$	
$c = 0$	$2\left(\frac{1 - z^p}{1 - z}\right)^2$
$0 < c < p/3$	$\frac{2 - z^{p-3c} - 2z^p - z^{p+3c} + 2z^{2p}}{(1 - z)^2}$
$p/3 < c < p/2$	$\frac{2 - z^{-p+3c} - z^{3p-3c} - z^{p+3c} - z^{5p-3c} + 2z^{4p}}{(1 - z)^2}$
$p/2 < c < 2p/3$	$\frac{2 - z^{-3c+2p} - z^{3c} - z^{-3c+4p} - z^{3c+2p} + 2z^{4p}}{(1 - z)^2}$
$2p/3 < c < p$	$\frac{2 - z^{3c-2p} - 2z^p - z^{4p-3c} + 2z^{2p}}{(1 - z)^2}$

In all cases, the singular vectors are known explicitly and are calculated by us for c in the range $p/2 < c < 2p/3$ and otherwise given by [Li14]. The character formulas are not provided by [Li14] but they are easily calculated from the singular vectors so we include them for completeness. When $\tau = \texttt{stand}$ we rely on the minor technical assumption 12.2.1 for c in the range $p/6 < c < p/3$, and the results for c in the range $p/3 < c < 2p/3$ are conjectural.

Proof. This is Theorem 12.0.1 in Chapter 12. For all τ and t, the case $c = 0$ is standard; when $t = 0$, the result is explained in Proposition 2.6.11, while for $t = 1$ the result follows from Proposition 2.6.13 and Corollary 7.2.6.

For $\tau = \texttt{triv}$, the remaining cases of $t = 1$, $c \neq 0 \in \mathbb{F}_p$ fall into several cases depending on where c lies in the set $\{0, 1, \ldots, p-1\}$. The paper [Li14] deals with all these intervals except one, $p/2 < c < 2p/3$, where they give conjectured degrees of the generators. The work of [Li14] can be found in Section 5.2 and we deal with the remaining case in Section 12.1.

For $\tau = \texttt{sign}$, the character formulas follow from the character formulas for $\texttt{triv}$ by Corollary 2.8.3.

For $\tau = \texttt{stand}$ $t = 1$ and $c \in \mathbb{F}_p$, the case $0 < c < p/3$ is done with all the proofs in Section 12.2, the case $p/3 < c < p/2$ is stated conjecturally and with no proofs at the end of that section. The cases $p/2 < c < p$ follow from them using Corollary 2.8.3 and the fact that $\texttt{stand} \otimes \texttt{sign} \cong \texttt{stand}$. ☐

The thesis is organised as follows:

- In Chapter 1 we give a basic description of reflection groups and the representation theory of the symmetric groups S_2 and S_3 in positive characteristic.

- In Chapter 2 we cover the definition of the rational Cherednik algebra, and give a detailed description of its representation theory in positive characteristic. We also describe a correspondence between certain irreducible representations which can reduce the number of cases we need to calculate.

- In Chapter 3 we discuss the symmetric group as a reflection group and its reflection representations in positive characteristic. We also describe the symmetric invariants.

- In Chapter 4 we specialise earlier concepts to the specific case of the rational Cherednik algebra $H_{t,c}(S_n, \mathfrak{h})$ in positive characteristic. We also explore how the representation theory of $H_{t,c}(S_n, \mathfrak{h})$ in our conventions differs from other authors. We explain how, in certain characteristic, we can translate results between these conventions and show that in this case the representation theory of $H_{t,c}(S_n, \mathfrak{h})$ does not depend on the choice of convention.

- In Chapter 5 we review similar work from the literature and explain how this work compares with ours. We also briefly mention other adjacent work in the literature.

- In Chapter 6 we prove Theorem 1 and describe the irreducible representations of $H_{t,c}(S_2, \mathfrak{h})$ in positive characteristic, for all p, t, and c.

- In Chapter 7 we describe a particular basis of Verma modules for $H_{t,c}(S_3, \mathfrak{h})$ in positive characteristic. The results in this chapter are used extensively in the rest of the thesis, as we perform many calculations in that basis.

- In Chapter 8 we prove Theorem 2 and describe the irreducible representations of $H_{t,c}(S_3, \mathfrak{h})$ in characteristic 2, for all t and c.
- In Chapter 9 we prove Theorem 3 and describe the irreducible representations of $H_{t,c}(S_3, \mathfrak{h})$ in characteristic 3, for all t and c.
- In Chapter 10 we explain how to reduce the space in which we need to look for singular vectors, and compile useful identities to be used in later computations.
- In Chapter 11 we prove Theorem 4 and describe the irreducible representations of $H_{t,c}(S_3, \mathfrak{h})$ in characteristic $p > 3$ for all t and generic values of c.
- In Chapter 12 we prove Theorem 5 and describe the irreducible representations of $H_{t,c}(S_3, \mathfrak{h})$ in characteristic $p > 3$ for all t special values of c. In particular when $\tau = \texttt{triv}, \texttt{sign}$ we take c in the range $p/2 < c < 2p/3$ to solve a previously unknown case. For $\tau = \texttt{stand}$ we take c in the range $0 < c < p/3$ and calculate the irreducible module, which also tells us the result in the range $2p/3 < c < p$. We conjecture the result for c in the range $p/3 < c < 2p/3$.
- Appendix A contains [Magma] code we wrote in order to compute examples and gain insight on the problem in general.

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Chapter 1

Preliminaries

1.1 Reflection groups

In order to construct a rational Cherednik algebra, we must choose a finite reflection group. Reflections are invertible linear maps which fix hyperplanes, and reflection groups are groups generated by such maps.

Let $\mathfrak{h}$ be a finite-dimensional vector space over an algebraically closed field $\Bbbk$.

Definition 1.1.1. A *hyperplane* in $\mathfrak{h}$ is a subspace of $\mathfrak{h}$ with dimension $\dim \mathfrak{h} - 1$.

Equivalently we say that hyperplanes have codimension 1. Let $GL(\mathfrak{h})$ be the group of invertible linear endomorphisms on $\mathfrak{h}$ and for $g \in GL(\mathfrak{h})$ denote by $g.y$ the action of g on $y \in \mathfrak{h}$ as a $\Bbbk[GL(\mathfrak{h})]$-module. Let $\mathrm{Fix}_{\mathfrak{h}}(g) = \{y \in \mathfrak{h} \mid g.y = y\}$ be the subspace of $\mathfrak{h}$ consisting of elements which are unchanged by the action of g.

Definition 1.1.2. A *reflection* is an element $s \in GL(\mathfrak{h})$ such that $\dim \mathrm{Fix}_{\mathfrak{h}}(s) = \dim \mathfrak{h} - 1$.

In other words, $s \in GL(\mathfrak{h})$ is a reflection if it fixes a hyperplane in $\mathfrak{h}$. If $g \in GL(\mathfrak{h})$ then we have

$$y \in \mathrm{Fix}_{\mathfrak{h}}(g) \iff g.y = y \iff y - g.y = (1-g).y = \mathbf{0} \iff y \in \ker_{\mathfrak{h}}(1-g)$$

which shows that $\mathrm{Fix}_{\mathfrak{h}}(g) = \ker_{\mathfrak{h}}(1-g)$. If additionally g is a reflection then

$$\dim \ker_{\mathfrak{h}}(1-g) = \dim \mathfrak{h} - 1.$$

By the rank-nullity theorem, when g is a reflection we get $\dim \mathrm{im}_{\mathfrak{h}}(1-g) = 1$. Therefore $s \in GL(\mathfrak{h})$ is a reflection if and only if $\mathrm{rank}_{\mathfrak{h}}(1-s) = 1$.

Example 1.1.3. Let $\mathfrak{h} = \mathbb{R}^2$ with the standard basis $y_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $y_2 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, and consider the invertible linear transformation $s \in GL(\mathfrak{h})$ defined on the basis by $s.y_1 = y_2$ and $s.y_2 = y_1$.

This linear transformation can be written as a matrix $s = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ with respect to the basis $\{y_1, y_2\}$ of $\mathfrak{h}$. Since $(1-s) = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]$ is a matrix of rank 1, s is a reflection. This example is illustrated in Figure 1.1.A.

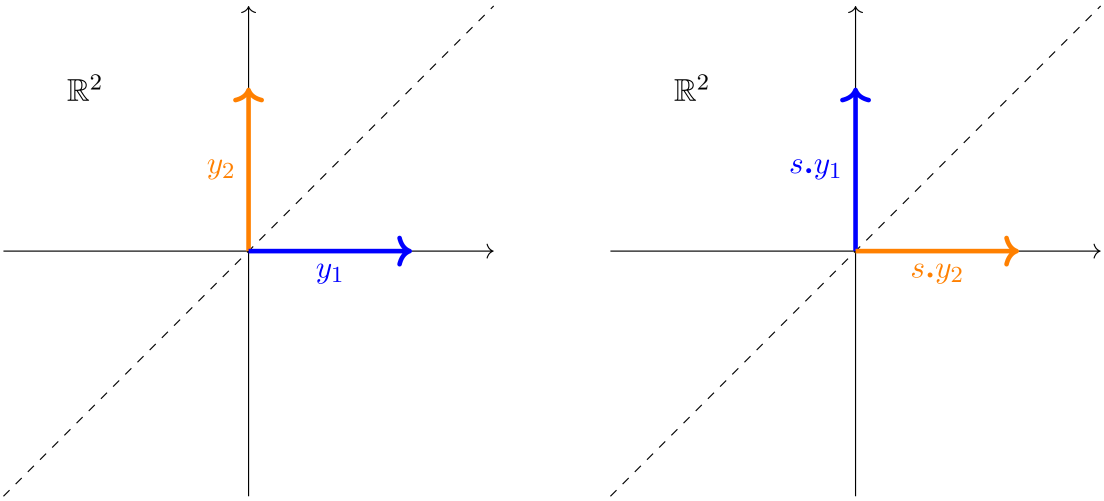


Figure 1.1.A: Diagram of a real reflection from Example 1.1.3. Points along the dashed line are fixed by s and the rest of the plane is reflected across that axis.

Example 1.1.4. Let $\mathfrak{h} = \mathbb{C}^n$ and let $\xi \in \mathbb{C}$ be an m^{th} root of unity for some positive integers $n, m \in \mathbb{Z}_{>0}$ and $\xi \neq 1$. Consider the $n \times n$ diagonal matrix

$$s = \begin{bmatrix} \xi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

as an element of $GL(\mathfrak{h})$. The matrix

$$1 - s = \begin{bmatrix} 1-\xi & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

has rank 1, therefore s is a reflection on $\mathfrak{h}$.

Example 1.1.5. Let $\mathfrak{h} = \mathbb{k}^2$ and consider the matrix $s = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in GL(\mathfrak{h})$. The matrix $1 - s = \left[\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right]$ has rank 1, therefore s is a reflection on $\mathfrak{h}$. It in important to note that $s^m = \left[\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right]$ therefore s has infinite order when $\mathbb{k}$ has characteristic 0.

Definition 1.1.6. A finite group $G \leq GL(\mathfrak{h})$ is a *finite reflection group* if the set

$$S = \{s \mid s \in G \text{ is a reflection}\}$$

of reflections in G generates the group.

Since some reflections have infinite order, it is necessary when considering finite reflection groups to ensure we restrict our attention to reflections of finite order. The reflection in Example 1.1.5 belongs to a finite reflection group only if the characteristic of $\Bbbk$ is positive.

Let $G \leq GL(\mathfrak{h})$ be a finite reflection group so that $\mathfrak{h}$ has the structure of a $\Bbbk[G]$-module. We call $\mathfrak{h}$ a *reflection representation* of G. We will now consider the dual representation of $\mathfrak{h}$ and we will see that reflections on $\mathfrak{h}$ coincide with reflections on the dual space.
Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$ and denote by $\langle \cdot, \cdot \rangle$ the canonical pairing $\mathfrak{h}^* \otimes \mathfrak{h} \to \Bbbk$. For any $g \in G$ and $x \in \mathfrak{h}^*$ the dual space has the structure of a $\Bbbk[G]$-module (the dual representation) with the action $g.x$ defined by $\langle g.x, y \rangle = \langle x, g^{-1}.y \rangle$ for every $y \in \mathfrak{h}$.

Proposition 1.1.7. *For every reflection $s \in G$ there exists a unique element $\alpha_s \otimes \alpha_s^\vee \in \mathfrak{h}^* \otimes \mathfrak{h}$ such that*

$$s.x = x - \langle x, \alpha_s^\vee \rangle \alpha_s \qquad\qquad s.y = y + \frac{\langle \alpha_s, y \rangle}{1 - \langle \alpha_s, \alpha_s^\vee \rangle} \alpha_s^\vee$$

for every $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$.

Proof. Let $s \in G$. First suppose that s acts as a reflection on the dual representation and therefore fixes a hyperplane of $\mathfrak{h}^*$. Since $\dim \operatorname{im}_{\mathfrak{h}^*}(1-s) = 1$ we may choose any nonzero vector in $\operatorname{im}_{\mathfrak{h}^*}(1-s)$ to be a basis.
If we choose a basis α_s of $\operatorname{im}_{\mathfrak{h}^*}(1-s)$ then for any $x \in \mathfrak{h}^*$ we can write $(1-s).x$ as a scalar multiple of α_s. Let $\alpha_s^\vee : \mathfrak{h}^* \to \Bbbk$ be the map satisfying $(1-s).x = \alpha_s^\vee(x)\alpha_s$ for all $x \in \mathfrak{h}$. The map $\alpha_s^\vee$ is a linear functional on $\mathfrak{h}^*$ and can therefore be thought of as an element of $\mathfrak{h}$. Note that $\alpha_s^\vee$ depends on the choice of α_s and a different choice of a basis $\widetilde{\alpha_s}$ would satisfy $\widetilde{\alpha_s} = \lambda \alpha_s$ for some nonzero constant $\lambda \in \Bbbk$ with the corresponding $\widetilde{\alpha_s}^\vee = \lambda^{-1}\alpha_s^\vee$. However, the tensor $\alpha_s \otimes \alpha_s^\vee$ does not depend on this choice because $\widetilde{\alpha_s} \otimes \widetilde{\alpha_s}^\vee = \alpha_s \otimes \alpha_s^\vee$. Therefore $\alpha_s \otimes \alpha_s^\vee \in \mathfrak{h}^* \otimes \mathfrak{h}$ is uniquely determined and satisfies $(1-s).x = \langle x, \alpha_s^\vee \rangle \alpha_s$ for every $x \in \mathfrak{h}$. We can rearrange this formula to obtain

$$s.x = x - \langle x, \alpha_s^\vee \rangle \alpha_s$$

for all $x \in \mathfrak{h}$, where α_s and $\alpha_s^\vee$ are both nonzero.
We can now substitute $x = \alpha_s$ and yield

$$s.\alpha_s = \alpha_s - \langle \alpha_s, \alpha_s^\vee \rangle \alpha_s = (1 - \langle \alpha_s, \alpha_s^\vee \rangle)\alpha_s$$

which shows that α_s is an eigenvector of s with eigenvalue $1 - \langle \alpha_s, \alpha_s^\vee \rangle$. Since s is invertible we can conclude that the eigenvalue $1 - \langle \alpha_s, \alpha_s^\vee \rangle$ is nonzero, and that the value of s^{-1} on α_s is given by the reciprocal

$$(s^{-1}).\alpha_s = \frac{1}{1 - \langle \alpha_s, \alpha_s^\vee \rangle} \alpha_s.$$

We may now calculate the action of s^{-1} on $x \in \mathfrak{h}^*$.

$$\begin{aligned} s.x = x - \langle x, \alpha_s^\vee \rangle \alpha_s &\iff x = (s^{-1}).x - \langle x, \alpha_s^\vee \rangle\, (s^{-1}).\alpha_s \\ &\iff (s^{-1}).x = x + \langle x, \alpha_s^\vee \rangle (s^{-1}).\alpha_s \end{aligned}$$

Using this formula and the definition of the dual representation, we can calculate the action of s on $y \in \mathfrak{h}$ in terms of α_s and $\alpha_s^\vee$ as follows.

$$\begin{aligned} \langle x,\ s.y \rangle &= \langle (s^{-1}).x,\ y \rangle \\ &= \big\langle x + \langle x, \alpha_s^\vee \rangle (s^{-1}).\alpha_s,\ y \big\rangle \\ &= \langle x,\ y \rangle + \langle x, \alpha_s^\vee \rangle \big\langle (s^{-1}).\alpha_s,\ y \big\rangle \\ &= \langle x,\ y \rangle + \langle x, \alpha_s^\vee \rangle \left\langle \frac{1}{1 - \langle \alpha_s, \alpha_s^\vee \rangle}\, \alpha_s,\ y \right\rangle \\ &= \langle x,\ y \rangle + \langle x, \alpha_s^\vee \rangle \frac{\langle \alpha_s, y \rangle}{1 - \langle \alpha_s, \alpha_s^\vee \rangle} \\ &= \langle x,\ y \rangle + \left\langle x,\ \frac{\langle \alpha_s, y \rangle}{1 - \langle \alpha_s, \alpha_s^\vee \rangle} \alpha_s^\vee \right\rangle \\ &= \left\langle x,\ y\ + \frac{\langle \alpha_s, y \rangle}{1 - \langle \alpha_s, \alpha_s^\vee \rangle} \alpha_s^\vee \right\rangle \end{aligned}$$

This holds for all $x \in \mathfrak{h}^*$ and therefore

$$s.y = y + \frac{\langle \alpha_s, y \rangle}{1 - \langle \alpha_s, \alpha_s^\vee \rangle} \alpha_s^\vee$$

for any $y \in \mathfrak{h}$. We can see that for any $y \in \mathfrak{h}$ the image $(1 - s).y$ is spanned by $\alpha_s^\vee$ therefore $\mathrm{rank}_{\mathfrak{h}}(1 - s) = 1$ and $s \in G$ is a reflection on $\mathfrak{h}$.
Finally, noting that $(\mathfrak{h}^*)^* \cong \mathfrak{h}$ as a representation we can conclude that $s \in G$ acts as a reflection on $\mathfrak{h}$ if and only if it acts as a reflection on $\mathfrak{h}^*$. $\square$

For a reflection $s \in GL(\mathfrak{h})$ there is a relation between the hyperplane $\mathrm{Fix}_{\mathfrak{h}}(s)$ and the element $\alpha_s \in \mathfrak{h}^*$ as seen by the following.

$$\begin{aligned} \mathrm{Fix}_{\mathfrak{h}}(s) &= \big\{ y \in \mathfrak{h} \mid y = s.y \big\} \\ &= \left\{ y \in \mathfrak{h} \;\middle|\; y = y + \frac{\langle \alpha_s, y \rangle}{1 - \langle \alpha_s, \alpha_s^\vee \rangle} \alpha_s^\vee \right\} \\ &= \left\{ y \in \mathfrak{h} \;\middle|\; \mathbf{0} = \frac{\langle \alpha_s, y \rangle}{1 - \langle \alpha_s, \alpha_s^\vee \rangle} \alpha_s^\vee \right\} \end{aligned}$$

Both $\alpha_s^\vee$ and $1 - \langle \alpha_s, \alpha_s^\vee \rangle$ are nonzero, therefore

$$\frac{\langle \alpha_s, y \rangle}{1 - \langle \alpha_s, \alpha_s^\vee \rangle} \alpha_s^\vee = \mathbf{0} \iff y \in \ker(\alpha_s)$$

and so $\mathrm{Fix}_{\mathfrak{h}}(s) = \ker(\alpha_s)$. Similarly, it can be shown that $\mathrm{Fix}_{\mathfrak{h}^*}(s) = \ker(\alpha_s^\vee)$.

Example 1.1.8. Let $\mathfrak{h} = \mathbb{R}^3$ with standard basis $\left\{ y_1 = \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right],\ y_2 = \left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right],\ y_3 = \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right] \right\}$. The dual space $\mathfrak{h}^*$ has dual basis $\{x_1 = [\,1\ 0\ 0\,],\ x_2 = [\,0\ 1\ 0\,],\ x_3 = [\,0\ 0\ 1\,]\}$. Let $s \in GL(\mathfrak{h})$ be the invertible linear transformation given by its action on the basis $s.y_1 = -y_1$, $s.y_2 = y_2$ and $s.y_3 = y_3$. As a matrix, s can be written as

$$[s]_{\mathfrak{h}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with respect to the basis $\{y_1, y_2, y_3\}$ of $\mathfrak{h}$. Since

$$[1-s]_{\mathfrak{h}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank 1, s is a reflection. The matrix $[s]_{\mathfrak{h}}{}^2$ is the identity matrix so s is a reflection of finite order and the group G generated by s is a finite reflection group. The subspace $\mathrm{Fix}_{\mathfrak{h}}(s) = \ker_{\mathfrak{h}}(1-s)$ is spanned by $\{y_2, y_3\}$, which is a hyperplane in $\mathbb{R}^3$, and $\mathrm{im}_{\mathfrak{h}}(1-s)$ is spanned by y_1. The action of s on the dual representation $\mathfrak{h}^*$ is given by the matrix

$$[s]_{\mathfrak{h}^*} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with respect to the dual basis $\{x_1, x_2, x_3\}$ of $\mathfrak{h}^*$. Furthermore

$$[1-s]_{\mathfrak{h}^*} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\mathrm{im}_{\mathfrak{h}^*}(1-s)$ is spanned by $\{x_1\}$ and we can choose a basis $\alpha_s = x_1$. Since α_s is an eigenvector of s with eigenvalue $1 - \langle \alpha_s, \alpha_s^\vee \rangle$, we compute $s.\alpha_s = s.x_1 = -x_1 = -\alpha_s$. The eigenvalue is $1 - \langle \alpha_s, \alpha_s^\vee \rangle = -1 \implies \langle \alpha_s, \alpha_s^\vee \rangle = 2$. Substituting our value of $\alpha_s = x_1$ we get $\langle x_1, \alpha_s^\vee \rangle = 2$, hence $\alpha_s^\vee = 2y_1 \in \mathrm{im}_{\mathfrak{h}}(1-s)$. Therefore $\alpha_s \otimes \alpha_s^\vee = 2x_1 \otimes y_1$. We could have made a different choice for the basis of $\mathrm{im}_{\mathfrak{h}^*}(1-s)$, say $\widetilde{\alpha_s} = 2x_1$. However the eigenvalue is still -1 and so $\langle \widetilde{\alpha_s}, \widetilde{\alpha_s}^\vee \rangle = 2 \implies \langle 2x_1, \widetilde{\alpha_s}^\vee \rangle = 2$, hence $\widetilde{\alpha_s}^\vee = y_1 \in \mathrm{im}_{\mathfrak{h}}(1-s)$. Once more, we get $\widetilde{\alpha_s} \otimes \widetilde{\alpha_s}^\vee = 2x_1 \otimes y_1 = \alpha_s \otimes \alpha_s^\vee$ illustrating that this choice is determined only up to mutual scaling. The formula for the action of s on $x = [\,a\ b\ c\,] \in \mathfrak{h}^*$ is

$$s.x = x - 2\langle x, y_1 \rangle x_1 = [\,a\ b\ c\,] - 2\,[\,a\ 0\ 0\,] = [\,-a\ b\ c\,]$$

and for $y = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] \in \mathfrak{h}$ we get

$$s.y = y - 2\langle x_1, y\rangle y_1 = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] - 2\left[\begin{smallmatrix} u \\ 0 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -u \\ v \\ w \end{smallmatrix}\right].$$

This example is illustrated in Figure 1.1.B.

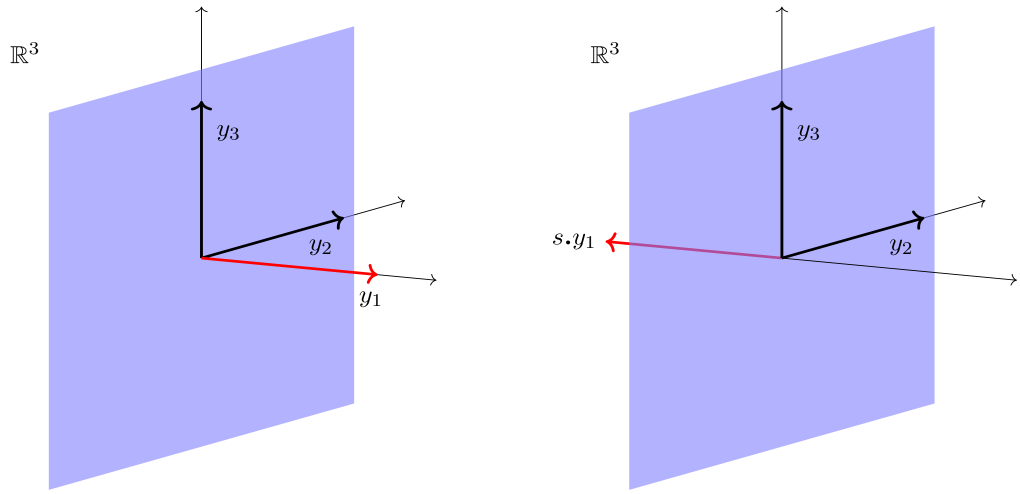


Figure 1.1.B: Diagram of a real reflection in $\mathbb{R}^3$ from Example 1.1.8. The shaded area represents the hyperplane $\mathrm{Fix}_{\mathfrak{h}}(s) = \ker(1-s) = \mathrm{span}\{y_2, y_3\} = \ker(x_1)$.

Example 1.1.9. Let ξ be a primitive m^{th} root of unity, for some positive integer m. Let $\mathfrak{h} = \mathbb{C}^3$ with basis $\{y_1, y_2, y_3\}$ and let $\mathfrak{h}^*$ be the dual space with dual basis $\{x_1, x_2, x_3\}$. Consider the element

$$[s]_{\mathfrak{h}} = \begin{bmatrix} \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in $GL(\mathfrak{h})$ written as a matrix with respect to the basis $\{y_1, y_2, y_3\}$ of $\mathfrak{h}$. The eigenvalues of s are readily seen from this diagonal matrix; there are $2 = \dim\mathfrak{h} - 1$ eigenvalues of 1, and an eigenvalue ξ not equal to 1. This implies that there is a 2-dimensional subspace of $\mathfrak{h}$ on which s acts as the identity. Indeed, the matrix

$$[1-s]_{\mathfrak{h}} = \begin{bmatrix} 1-\xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank 1, so s is a reflection and must therefore fix a hyperplane. As in Example 1.1.8, s has finite order so the group G generated by s is a finite reflection group, which is cyclic of order m. Since $\ker_{\mathfrak{h}}(1-s)$ is spanned by $\{y_2, y_3\}$, and $\ker(\alpha_s) = \ker_{\mathfrak{h}}(1-s)$, choose $\alpha_s = x_1$. Now $\alpha_s^\vee$ is a basis for $\mathrm{im}_{\mathfrak{h}}(1-s)$ so $\alpha_s^\vee$ must be a multiple of y_1. The eigenvalue ξ satisfies $\xi = 1 - \langle \alpha_s, \alpha_s^\vee\rangle$

which we rearrange to give $\langle \alpha_s, \alpha_s^\vee \rangle = 1 - \xi$. Therefore $\langle x_1, \alpha_s^\vee \rangle = 1 - \xi \implies \alpha_s^\vee = (1-\xi)y_1$, and $\alpha_s \otimes \alpha_s^\vee = (1-\xi)x_1 \otimes y_1$. For $x = \left[\begin{smallmatrix} a & b & c \end{smallmatrix}\right] \in \mathfrak{h}^*$ and $y = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] \in \mathfrak{h}$ the action of s is given by the formulas

$$s.x = x - (1-\xi)\langle x, y_1 \rangle x_1 = \left[\begin{smallmatrix} a & b & c \end{smallmatrix}\right] - (1-\xi)\left[\begin{smallmatrix} a & 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \xi a & b & c \end{smallmatrix}\right]$$

and

$$s.y = y + (\xi^{-1} - 1)\langle x_1, y \rangle y_1 = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] + (\xi^{-1} - 1)\left[\begin{smallmatrix} u \\ 0 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \xi^{-1}u \\ v \\ w \end{smallmatrix}\right].$$

Example 1.1.10. Let the characteristic of $\Bbbk$ be $p > 0$. Let $\mathfrak{h} = \Bbbk^3$ with basis $\{y_1, y_2, y_3\}$ and let $\mathfrak{h}^*$ be the dual space with dual basis $\{x_1, x_2, x_3\}$. Consider the element

$$[s]_{\mathfrak{h}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in $GL(\mathfrak{h})$ written as a matrix with respect to the basis $\{y_1, y_2, y_3\}$ of $\mathfrak{h}$. This matrix has two eigenvalues which are 1, and a generalised eigenvalue which is also 1. Since

$$[1-s]_{\mathfrak{h}} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank 1, s is a reflection. The m^{th} power of s is given by the matrix

$$s^m = \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

therefore s^p is the identity matrix so s has finite order, and the group generated by s is a finite reflection group. A reflection of this kind which is not diagonalisable is called a *transvection.* Finite reflection groups containing transvections occur only when the characteristic of $\Bbbk$ is positive, because if $\Bbbk$ had characteristic 0 then the group generated by s would not be finite. Since $\mathrm{im}_{\mathfrak{h}}(1-s)$ is spanned by $\{y_1\}$, we know that $\alpha_s^\vee$ must be a multiple of y_1. Furthermore, $\ker_{\mathfrak{h}}(1-s)$ is spanned by $\{y_1, y_3\}$ and $\ker(\alpha_s) = \ker_{\mathfrak{h}}(1-s)$ therefore we choose $\alpha_s = x_2$. The action of s on y_2 is given by

$$s.y_2 = y_2 + \frac{\langle x_2, y_2 \rangle}{1 - \langle x_2, \alpha_s^\vee \rangle}\alpha_s^\vee = y_1 + y_2,$$

therefore

$$\frac{1}{1 - \langle x_2, \alpha_s^\vee \rangle}\alpha_s^\vee = y_1$$

which is satisfied by $\alpha_s^\vee = y_1$, hence $\alpha_s \otimes \alpha_s^\vee = x_2 \otimes y_1$. Note that in this instance $\langle \alpha_s, \alpha_s^\vee \rangle = 0$. We can deduce that

$$\ker_{\mathfrak{h}^*}(1-s) = \ker(\alpha_s^\vee) = \ker(y_1) = \operatorname{span}\{x_2, x_3\}$$

and

$$\operatorname{im}_{\mathfrak{h}^*}(1-s) = \operatorname{span}\{\alpha_s\} = \operatorname{span}\{x_2\}.$$

Given $x = \left[\begin{smallmatrix} a & b & c \end{smallmatrix}\right] \in \mathfrak{h}^*$ and $y = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] \in \mathfrak{h}$, the formulas for the action of s are given by

$$s.x = x - \langle x, y_1 \rangle x_2 = \left[\begin{smallmatrix} a & b & c \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 & a & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b-a & c \end{smallmatrix}\right]$$

and

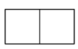
$$s.y = y + \frac{\langle x_2, y \rangle}{1 - \langle x_2, y_1 \rangle} y_1 = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] + \left[\begin{smallmatrix} v \\ 0 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} u+v \\ v \\ w \end{smallmatrix}\right]$$

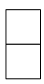
These are the 3 main types of examples of reflections. *Real* reflections are elements of order 2 with an eigenvalue of -1, and they match our intuition for the meaning of reflection. *Complex* reflections have finite order and an eigenvalue which is a root of unity not equal to one. *Transvections* are reflections which are non-diagonalisable, they are unique to finite reflection groups in the case of positive characteristic.

1.2 Representation theory of S_2 and S_3

Representations of rational Cherednik algebras can be induced from representations of finite reflection groups. The representation theory of the finite reflection group associated with a Cherednik algebra therefore informs the representation theory of the Cherednik algebra. Furthermore, irreducible representations of a rational Cherednik algebra are parametrised by irreducible representations of its associated reflection group.
Representations of S_n are labelled by partitions of n and we can use Young diagrams to represent the partitions. In this section, we will describe the representation theory of the symmetric groups S_2 and S_3 in any characteristic and fix some notation. Most importantly we will outline which representations are irreducible in positive characteristics. For more detail on the results of this chapter, see Chapter 10 of [We16].

We will first consider the representation theory of S_2. What follows are the Young diagrams for the partitions of 2.

(2) 

(1,1) 

The partition (2) corresponds to triv, the trivial representation of S_2 with the basis $\{1_{\texttt{triv}}\}$ and group action given by $g.1_{\texttt{triv}} = 1_{\texttt{triv}}$ for all $g \in S_2$. The partition $(1,1)$ labels sign, the sign representation of S_2 with basis $\{1_{\texttt{sign}}\}$ and group action given by

$$g.1_{\texttt{sign}} = \begin{cases} 1_{\texttt{sign}} & g \text{ is an even permutation in } S_2 \\ -1_{\texttt{sign}} & g \text{ is an odd permutation in } S_2 \end{cases}$$

for all $g \in S_2$. The characters of these irreducible representations in characteristic 0 are shown in the following character table.

	(1)	(12)
triv	1	1
sign	1	-1

For a finite group G, let $\widehat{G}$ denote the set of irreducible representations of G over $\Bbbk$. In characteristic 0, we have $\widehat{S_2} = \{\texttt{triv}, \texttt{sign}\}$.

To detail the representation theory of symmetric groups in positive characteristic, we require the following definition.

Definition 1.2.1. Let p be a prime number. An element of a group is *p-regular* if its order is not a multiple of p.

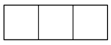
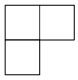
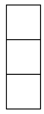
To obtain the character table of S_2 in characteristic p we keep only the columns representing conjugacy classes of p-regular elements. The result is called the *Brauer character* table. Brauer characters carry slightly less information than ordinary characters because, in finite characteristic, representations which are not isomorphic may have identical Brauer characters. Transpositions have order 2, therefore they are not 2-regular elements of S_2. By removing the column of transpositions from the ordinary character table, we obtain the Brauer character table in characteristic 2 as shown.

	(1)	(12)
triv	1	1
sign	1	-1

In characteristic 2, since $-1 \equiv 1$ we can deduce that triv and sign have identical S_2 actions and as a consequence their Brauer characters match. Hence in characteristic 2, there is only one irreducible representation and we have $\widehat{S_2} = \{\texttt{triv}\}$.

In any characteristic $p > 2$, the elements of S_2 are all p-regular therefore the Brauer character table looks the same as it does in characteristic 0 and $\widehat{S_2} = \{\texttt{triv}, \texttt{sign}\}$. Additionally, the category of finite-dimensional representations of S_2 in characteristic $p > 2$ is semisimple, as it is in characteristic 0.

We will now consider the representation theory of S_3. The partitions of 3 have Young diagrams

(3)  **(2,1)**  **(1,1,1)** 

which label trivial, standard, and sign representations of S_3. The representations `triv` and `sign` are defined similarly as they were for S_2. The partition (3) labels the representation `triv` which has a basis $\{1_{\texttt{triv}}\}$ and the group action is given by $g.1_{\texttt{triv}} = 1_{\texttt{triv}}$ for all $g \in S_3$. The partition $(1, 1, 1)$ labels the representation `sign` of which has a basis $\{1_{\texttt{sign}}\}$ and the group action is given by

$$g.1_{\texttt{sign}} = \begin{cases} 1_{\texttt{sign}} & g \text{ is an even permutation in } S_3 \\ -1_{\texttt{sign}} & g \text{ is an odd permutation in } S_3 \end{cases}$$

for all $g \in S_3$. To define `stand`, first let V be the permutation representation of S_3 with basis $\{y_1, y_2, y_3\}$ and the group action given by $g.y_i = y_{g(i)}$ for all $g \in S_3$. Now `stand` is defined as the subrepresentation

$$\{a_1y_1 + a_2y_3 + a_3y_3 \in V \mid a_1 + a_2 + a_3 = 0\}$$

of V. For more information about this representation see Chapter 3. The characters of these representations are shown in the following ordinary character table.

	(1)	(12)	(123)
`triv`	1	1	1
`sign`	1	-1	1
`stand`	2	0	-1

and we have $\widehat{S_3} = \{\texttt{triv}, \texttt{sign}, \texttt{stand}\}$ in characteristic 0.

To obtain the Brauer character table of S_3 in characteristic 2, we keep the column of 3-cycles which are 2-regular and remove the column of transpositions which are not 2-regular.

	(1)	(12)	(123)
`triv`	1	1	1
`sign`	1	-1	1
`stand`	2	0	-1

In a similar way to what we saw for S_2, the representations `triv` and `sign` are the same representation of S_3 in characteristic 2 and therefore have the same Brauer characters. Also, the representation `stand` remains irreducible in characteristic 2 (for proof, see Lemma 3.1.4). Hence $\widehat{S_2} = \{\texttt{triv}, \texttt{stand}\}$ in characteristic 2.

Next we consider characteristic 3. In S_3 the 3-cycles have order 3, so the 3-cycles are not 3-regular. However transpositions have order 2 and therefore are 3-regular in S_3. The following is the Brauer character table of S_3 in characteristic 3.

	(1)	(12)	(123)
`triv`	1	1	1
`sign`	1	-1	1
`stand`	2	0	-1

Here we see that `triv` and `sign` have linearly independent characters and are both irreducible because they are 1-dimensional. We also see that the Brauer character of `stand` is equal to the sum of the Brauer characters of `triv` and `sign`. This happens because there is a short exact sequence of S_3 representations

$$0 \to \texttt{triv} \to \texttt{stand} \to \texttt{sign} \to 0$$

which shows that in characteristic 3, `stand` is not irreducible. The copy of `triv` inside `stand` is spanned by the vector $y_1 + y_2 + y_3$, whose coefficients sum to zero in characteristic 3. Therefore we have $\widehat{S_3} = \{\texttt{triv}, \texttt{sign}\}$ in characteristic 3.

Elements of S_3 have order 1, 2, or 3. Therefore in all other characteristics besides 2 and 3 the Brauer character table is identical to the ordinary character table, and the category of finite-dimensional representations is semisimple. Therefore in characteristic $p > 3$ we have $\widehat{S_3} = \{\texttt{triv}, \texttt{sign}, \texttt{stand}\}$

Chapter 2

Rational Cherednik Algebras and Representation Theory

2.1 Rational Cherednik algebras

In 1992, the double affine Hecke algebras were defined by Ivan Cherednik [Ch92] in order to study quantum Knizhnik–Zamolodchikov equations. In 1995, Cherednik used double affine Hecke algebras to prove the Macdonald constant term conjecture [Ch95].
In 2002, [EtGi02] Pavel Ilyich Etingof and Victor Ginzburg defined "*certain 'rational' degenerations of the double affine Hecke algebra introduced earlier by Cherednik.*" Today, these rational degenerations of double affine Hecke algebras are better known as Cherednik algebras. The rational Cherednik algebras and their representation theory will be explored in this chapter. We follow [BaCh13a] in our definitions, but overviews of this material are also available in [Go10, EtMa11, Be12].

Let $\Bbbk$ be an algebraically closed field. Let G be a finite reflection group over $\Bbbk$ with reflection representation $\mathfrak{h}$ and dual representation $\mathfrak{h}^*$. Let $S \subset G$ be the set of reflections in G and choose a function $c : S \to \Bbbk$, $s \mapsto c(s) = c_s$ with the conjugation invariant property $c(gsg^{-1}) = c(s)$ for all $g \in G$ and $s \in S$. Denote by $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ the *tensor algebra* of $\mathfrak{h} \oplus \mathfrak{h}^*$, and fix a constant $t \in \Bbbk$.

Definition 2.1.1. The *rational Cherednik algebra* $H_{t,c}(G, \mathfrak{h})$ is the quotient of the associative algebra $\Bbbk[G] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = \langle x, y\rangle t - \sum_{s \in S} c_s \langle (1 - s).x, y\rangle s$$

for all $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$.

Recall that for each $s \in S$ there is a unique element $\alpha_s \otimes \alpha_s^\vee \in \mathfrak{h}^* \otimes \mathfrak{h}$ as described in

Proposition 1.1.7. The constant $\langle (1-s).x, y\rangle$ can be written as

$$\langle (1-s).x, y\rangle = \big\langle \langle x, \alpha_s^\vee\rangle \alpha_s, y\big\rangle = \langle x, \alpha_s^\vee\rangle\langle \alpha_s, y\rangle$$

and this equivalent form is used by some authors when defining the commutator $[y, x]$. Using properties of the dual representation, we can also derive the equality

$$\langle (1-s).x, y\rangle = \langle x, (1-s^{-1}).y\rangle$$

which is useful in some calculations.

The rational Cherednik algebra is an associative infinite-dimensional algebra with a unit, and is noncommutative for sensible choices of parameters G, $\mathfrak{h}$, t and c.

Proposition 2.1.2. *For any $\lambda \in \Bbbk^\times$, there is an isomorphism of algebras $H_{t,c}(G,\mathfrak{h}) \cong H_{\lambda t,\lambda c}(G,\mathfrak{h})$.*

Proof. The map $\phi : H_{t,c}(G,\mathfrak{h}) \to H_{\lambda t,\lambda c}(G,\mathfrak{h})$, defined by $\phi(x) = \lambda x$, $\phi(y) = y$, and $\phi(g) = g$ for $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, and $g \in G$, is an isomorphism of algebras. □

In other words, rational Cherednik algebras are isomorphic up to simultaneous rescaling of parameters t and c by a nonzero constant. For this reason we can restrict our attention to just the cases of $t = 0$ and $t = 1$ individually.

Let $S(\mathfrak{h})$ and $S(\mathfrak{h}^*)$ denote the symmetric algebras on $\mathfrak{h}$ and $\mathfrak{h}^*$ respectively.

Theorem 2.1.3. *There is an isomorphism of vector spaces*

$$S(\mathfrak{h}^*) \otimes \Bbbk[G] \otimes S(\mathfrak{h}) \cong H_{t,c}(G,\mathfrak{h})$$

given by multiplication.

This theorem is called the Poincaré-Birkhoff-Witt (PBW) theorem. It is analogous to a result originally proved in the context of Lie algebras and it can be found in [EtGi02] for rational Cherednik algebras in characteristic 0, and in [BaCh13a] for rational Cherednik algebras in characteristic p.

If $\{y_1, \ldots, y_m\}$ is a basis for $\mathfrak{h}$ and $\{x_1, \ldots, x_m\}$ is a basis for $\mathfrak{h}^*$ then the PBW theorem gives the following set as a basis for $H_{t,c}(G,\mathfrak{h})$.

$$\left\{ x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} g y_1^{b_1} y_2^{b_2} \cdots y_m^{b_m} \mid a_i, b_j \geq 0,\ g \in G \right\}$$

By setting degree $x = 1$, degree $g = 0$, degree $y = -1$ for all $x \in \mathfrak{h}^*$, $g \in G$, and $y \in \mathfrak{h}$ we obtain a $\mathbb{Z}$-grading on the algebra.

2.2 Verma modules

The PBW theorem gives us a decomposition of the rational Cherednik algebra into three tensor factors. That decomposition is similar in structure to factorisation of the universal enveloping algebra arising from a triangular decomposition of a semisimple Lie algebra in characteristic 0, for which Bernstein, Gelfand, and Gelfand defined a category $\mathcal{O}$.
The similarity motivates a definition of category $\mathcal{O}$ for rational Cherednik algebras over fields of characteristic 0, where the subalgebra $\Bbbk[G]$ takes the place of the Cartan subalgebra and the subalgebras $S(\mathfrak{h}^*)$ and $S(\mathfrak{h})$ play the roles of positive and negative nilpotent subalgebras ([EtMa11] Section 3.5). As a consequence we obtain the familiar notion of Verma modules, which behave as standard objects in a highest-weight category with the expected properties. In characteristic p it is more complicated, however by carefully choosing our definitions in this setting we can replicate familiar ideas. For a detailed overview in characteristic 0, see [GGOR03]. In characteristic p we follow the approach of [BaCh13a].

Let $\tau \in \widehat{G}$ be an irreducible representation of G over $\Bbbk$ and enrich its structure to a $\Bbbk[G] \ltimes S(\mathfrak{h})$-module by allowing $\mathfrak{h}$ to act on τ by zero. Since $\Bbbk[G] \ltimes S(\mathfrak{h})$ is a subalgebra of $H_{t,c}(G, \mathfrak{h})$ we can induce the representation τ to a representation of the Cherednik algebra.

Definition 2.2.1. The *Verma module corresponding to* τ is the induced $H_{t,c}(G, \mathfrak{h})$-module

$$M_{t,c}(G, \mathfrak{h}, \tau) = \operatorname{Ind}_{\Bbbk[G]\ltimes S(\mathfrak{h})}^{H_{t,c}(G,\mathfrak{h})} \tau.$$

We will abbreviate notation to $M_{t,c}(\tau)$ if it is clear which algebra we are working with. From the definition, we have

$$M_{t,c}(\tau) = H_{t,c}(G, \mathfrak{h}) \bigotimes_{\Bbbk[G]\ltimes S(\mathfrak{h})} \tau$$

as an $H_{t,c}(G, \mathfrak{h})$-module, with action given by left multiplication. We can apply the PBW theorem to obtain an isomorphism of vector spaces,

$$M_{t,c}(\tau) \cong S(\mathfrak{h}^*) \otimes \tau.$$

The Verma module $M_{t,c}(\tau)$ inherits a $\mathbb{Z}$-grading from $H_{t,c}(G, \mathfrak{h})$ and by choosing that the graded piece $\Bbbk \otimes \tau$ is in degree zero we obtain an $\mathbb{N}_0$-grading on $M_{t,c}(\tau)$ which corresponds to the grading on $S(\mathfrak{h}^*)$ in $S(\mathfrak{h}^*) \otimes \tau$. We denote by $M^d_{t,c}(\tau)$ the graded piece of $M_{t,c}(\tau)$ with degree d. The action of the rational Cherednik algebra $H_{t,c}(G, \mathfrak{h})$ can be derived from the definition of the Verma module $M_{t,c}(\tau)$ with the following result. For any $f \otimes v \in S(\mathfrak{h}^*) \otimes \tau$,

$$x.(f \otimes v) = (xf) \otimes v,$$

$$g.(f \otimes v) = (g.f) \otimes (g.v),$$
$$y.(f \otimes v) = t\partial_y(f) \otimes v - \sum_{s \in S} c_s \langle \alpha_s, y \rangle \frac{(1-s).f}{\alpha_s} \otimes s.v$$

for all $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, $g \in G$. Given any $y \in \mathfrak{h}$ and $f, f' \in S(\mathfrak{h}^*)$ the partial derivative ∂_y is characterised by the product rule $\partial_y(ff') = \partial_y(f)f' + f\partial_y(f')$ and defined in degree 1 by $\partial_y(x) = \langle x, y \rangle$ for all $x \in \mathfrak{h}^*$. The differential–difference operators of the form

$$D_y = t\partial_y \otimes 1 - \sum_{s \in S} c_s \langle \alpha_s, y \rangle \frac{(1-s)}{\alpha_s} \otimes s$$

corresponding to the action of $y \in \mathfrak{h}$ on the Verma module are known as *Dunkl operators.* First defined in 1989 by Charles Francis Dunkl, the Dunkl operators are known to commute ([Du89], Theorem 1.9). Rational Cherednik algebras are therefore the algebras generated by a reflection group algebra, a polynomial algebra, and Dunkl operators.

Verma modules are a family of infinite-dimensional representations with finite-dimensional graded pieces. As usual, we can obtain simple modules from Verma modules by taking a quotient. However, the quotient we consider is different to the one usually considered in characteristic 0, as highlighted by the following proposition.

Proposition 2.2.2 ([BaCh13a], Corollary 2.21)**.** *Suppose the characteristic of* $\Bbbk$ *is positive. The Verma module* $M_{t,c}(\tau)$ *has a unique maximal proper graded submodule, denoted* $J_{t,c}(\tau)$*, and the quotient* $L_{t,c}(\tau) = M_{t,c}(\tau)/J_{t,c}(\tau)$ *is an irreducible* $H_{t,c}(G, \mathfrak{h})$*-module.*

In characteristic 0, at $t = 1$ all submodules of $M_{t,c}(\tau)$ are graded and that adjective could be omitted from the proposition without consequence. In characteristic p, however, for any t there are submodules which are not graded and the sum of all proper submodules is the whole Verma module. Therefore it is crucial in positive characteristic to consider only the quotients of Verma modules by graded submodules.

Our aim is to describe the simple modules $L_{t,c}(\tau)$ when the reflection group G is the symmetric group S_2 or S_3 in a field of positive characteristic p, as we vary the parameters t, c and τ.

2.3 The Casimir element

In the context of Lie algebras where Casimir elements were first defined, they are particular elements in the centre of the universal enveloping alegbra. They are a useful tool in the representation theory of Lie algebras because the action of Casimir elements can be used to classify some representations and distinguish blocks. In this section, we will define the Casimir element of a rational Cherednik algebra and detail some of its properties.

Let $\{y_1, \ldots, y_m\}$ be a basis for $\mathfrak{h}$ and let $\{x_1, \ldots, x_m\}$ be a dual basis for $\mathfrak{h}^*$.

Definition 2.3.1. The element

$$\Omega = \sum_{i=1}^{m} x_i y_i + \sum_{s \in S} c_s (1 - s)$$

is called the *Casimir element* of $H_{t,c}(G, \mathfrak{h})$.

The Casimir element consists of two sums, and the second of these sums commutes with elements of the group as shown by the following proposition.

Proposition 2.3.2. *The element $\sum_{s \in S} c_s(1 - s)$ is central in $\Bbbk[G]$.*

Proof. We can express this element as a difference of two sums,

$$\sum_{s \in S} c_s(1 - s) = \sum_{s \in S} c_s - \sum_{s \in S} c_s s.$$

The first sum $\sum_{s \in S} c_s$ is a constant, so it is central in $\Bbbk[G]$. We can group the sum $\sum_{s \in S} c_s s$ by conjugacy class. In any group algebra, the sum of all elements in the same conjugacy class forms a central element. Since the value of c_s is fixed across each conjugacy class, the sum $\sum_{s \in S} c_s s$ is a linear combination of central elements, and is therefore central. □

We will now prove some commutation relations involving the Casimir element. These properties will be used later in this section to prove statements concerning the action of the Casimir element on Verma modules.

Proposition 2.3.3. *The Casimir element Ω satisfies the following:*

1. $[\Omega, x] = tx$
2. $[\Omega, x_{i_1} x_{i_2} \cdots x_{i_d}] = dt x_{i_1} x_{i_2} \ldots x_{i_d}$
3. $[\Omega, y] = -ty$
4. $[\Omega, y_{i_1} y_{i_2} \cdots y_{i_d}] = -dt y_{i_1} y_{i_2} \cdots y_{i_d}$
5. $[\Omega, g] = 0$

for all $d \in \mathbb{N}$, $x, x_{i_1}, \ldots, x_{i_d} \in \mathfrak{h}^*$, $y, y_{i_1}, \ldots, y_{i_d} \in \mathfrak{h}$ and all $g \in G$.

Proof. We will first prove $[\Omega, x] = tx$. Let $x \in \mathfrak{h}^*$ be arbitrary.

$$\begin{aligned}[\Omega, x] &= \left(\sum_{i=1}^{m} x_i y_i + \sum_{s \in S} c_s (1 - s) \right) x - x \left(\sum_{i=1}^{m} x_i y_i + \sum_{s \in S} c_s (1 - s) \right) \\ &= \sum_{i=1}^{m} x_i [y_i, x] + \sum_{s \in S} c_s \big(xs - sx \big)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} x_i\Big(\langle x, y_i\rangle t - \sum_{s\in S} c_s\langle (1-s).x, y_i\rangle s\Big) + \sum_{s\in S} c_s\big(x - s.x\big)s \\
&= \sum_{i=1}^{m} x_i\langle x, y_i\rangle t - \sum_{s\in S} c_s \sum_{i=1}^{m}\langle (1-s).x, y_i\rangle x_i s + \sum_{s\in S} c_s\big((1-s).x\big)s \\
&= xt - \sum_{s\in S} c_s\big((1-s).x\big)s + \sum_{s\in S} c_s\big((1-s).x\big)s \\
&= tx
\end{aligned}
$$

The second statement can be proved by induction on d using the previous part as the base case, and the third and fourth statements are proved analogously.
Finally we show $[\Omega, g] = 0$. Let $g \in G$ be arbitrary.

$$
\begin{aligned}
[\Omega, g] &= \Big[\sum_{i=1}^{m} x_i y_i + \sum_{s\in S} c_s(1-s), g\Big] \\
&= \Big[\sum_{i=1}^{m} x_i y_i, g\Big] + \Big[\sum_{s\in S} c_s(1-s), g\Big]
\end{aligned}
$$

By Proposition 2.3.2, the second commutator is zero, so we must show that g commutes with the sum $\sum_{i=1}^{m} x_i y_i$.

$$
\begin{aligned}
\Big[\sum_{i=1}^{m} x_i y_i, g\Big] &= \sum_{i=1}^{m} x_i y_i g - \sum_{i=1}^{m} g x_i y_i \\
&= \sum_{i=1}^{m} x_i y_i g - \sum_{i=1}^{m} (g.x_i)(g.y_i) g \\
&= \Big(\sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} (g.x_i)(g.y_i)\Big) g
\end{aligned}
$$

Therefore we are done if

$$
\sum_{i=1}^{m} x_i y_i = \sum_{i=1}^{m} (g.x_i)(g.y_i)
$$

and this follows from the fact that $\mathfrak{h}$ and $\mathfrak{h}^*$ are dual representations and the bases $\{y_1, \ldots, y_m\}$ and $\{x_1, \ldots, x_m\}$ are dual to each other. □

When $t = 1$, we have $[\Omega, x] = x$, $[\Omega, g] = 0$, and $[\Omega, y] = -y$, for all $x \in \mathfrak{h}^*$, $g \in G$ and $y \in \mathfrak{h}$. Therefore, in characteristic 0, Ω can be used to grade rational Cherednik algebras in a way which matches the grading given at the end of Section 2.1. However, in characteristic p we have $[\Omega, x^p] = px^p = 0$ and therefore Ω only gives a grading modulo p.

We now consider the action of the Casimir element Ω on Verma modules, and deduce an important property.

Proposition 2.3.4. *Let τ be any irreducible representation of G and consider the Verma module $M_{t,c}(\tau)$. The Casimir element Ω acts on every element of degree zero in $M_{t,c}(\tau)$ by some constant $\Omega|_\tau$.*

Proof. Degree zero of the Verma module $M_{t,c}(\tau)$ is just the irreducible representation τ, so let $v \in \tau$ be arbitrary. The action of Ω on v is given by

$$\begin{aligned}\Omega.v &= \Big(\sum_{i=1}^{m} x_i y_i + \sum_{s\in S} c_s(1-s)\Big).v \\ &= \Big(\sum_{i=1}^{m} x_i y_i\Big).v + \Big(\sum_{s\in S} c_s(1-s)\Big).v\end{aligned}$$

The first term of the preceding expression is equal to zero, because all the $y_i \in \mathfrak{h}$ act by zero on τ by definition of the Verma module. By Proposition 2.3.2 and Schur's lemma, the sum $\sum_{s\in S} c_s(1-s)$ is central in $\mathbb{k}[G]$ and therefore acts on the irreducible representation τ as a multiple of the identity, so by some constant $\Omega|_\tau \in \mathbb{k}$. This constant can be calculated from the character χ_τ as

$$\Omega|_\tau = \sum_{s\in S} c_s - \frac{1}{(\dim\,\tau)} \sum_{s\in S} c_s \chi_\tau(s).$$

□

Corollary 2.3.5. *Suppose that $t = 1$. The Casimir element Ω acts on $M^d_{1,c}(\tau)$ by $(\Omega|_\tau + d)$.*

Proof. Consider an element of the form $x_{i_1}\cdots x_{i_d} \otimes v$, where each $x_{i_k} \in \{x_1, \dots, x_m\}$ is a basis vector for $\mathfrak{h}^*$, and $v \in \tau$. Such elements span the homogeneous component in $M^d_{1,c}(\tau)$. By Statement 2 of Proposition 2.3.3, the action of Ω on $x_{i_1}\cdots x_{i_d} \otimes v$ can be calculated as

$$\begin{aligned}\Omega.(x_{i_1}\cdots x_{i_d} \otimes v) &= \Omega x_{i_1}\cdots x_{i_d} \otimes v \\ &= \big(x_{i_1}\cdots x_{i_d}\Omega + [\Omega, x_{i_1}\cdots x_{i_d}]\big) \otimes v \\ &= x_{i_1}\cdots x_{i_d}\Omega \otimes v + d x_{i_1}\cdots x_{i_d} \otimes v \\ &= (\Omega|_\tau) x_{i_1}\cdots x_{i_d} \otimes v + d x_{i_1}\cdots x_{i_d} \otimes v \\ &= (\Omega|_\tau + d) x_{i_1}\cdots x_{i_d} \otimes v.\end{aligned}$$

□

This property of the Casimir is used in Lemma 8.4.8 and throughout Chapter 12.

2.4 Singular vectors

The concept of a singular vector arises in many areas of representation theory. We can construct simple modules of rational Cherednik algebras by taking the quotient of a Verma module by its maximal proper graded submodule and we will see that singular vectors aid in this task.

Definition 2.4.1. Let M be a $\mathbb{Z}$-graded $H_{t,c}(G,\mathfrak{h})$-module with degrees bounded from below, that is, there exists an minimum degree such that $M = \bigoplus_{d\geq l} M^d$ for some integer l. Let $d > l$ and let $f \in M^d$ be a homogeneous element of degree strictly larger than the minimum. If $y.f = 0$ for every $y \in \mathfrak{h}$, then f is called a *singular vector.*

Singular vectors are simultaneously in the kernel of every Dunkl operator D_y for all $y \in \mathfrak{h}$. The graded modules we consider are Verma modules and their quotients by graded submodules. Since Verma modules have minimum degree 0, a singular vector of a Verma module must be a homogeneous element with strictly positive degree.

Example 2.4.2. Suppose $t = 0$ and $c : S \to \Bbbk$ is defined by $c(s) = 0$ for all $s \in S$. Let us show that for any τ, the irreducible quotient of $M_{0,0}(\tau)$ is concentrated in degree 0, that is $L_{0,0}(\tau) \cong \tau$. When $t = 0$ and $c = 0$, the Dunkl operator D_y is identically zero for any $y \in \mathfrak{h}$ so every homogeneous vector with positive degree in $M_{0,0}(\tau)$ is singular. In particular, everything in $S^i(\mathfrak{h}^*) \otimes \tau$ with $i \geq 1$ is singular and $J_{0,0}(\tau) = \bigoplus_{i=1}^{\infty} S^i(\mathfrak{h}^*) \otimes \tau$ is the proper submodule consisting of everything above degree 0. This submodule is maximal because if it were any larger it would be the whole module. Therefore

$$\begin{aligned} L_{0,0}(\tau) &= M_{0,0}(\tau)/J_{0,0}(\tau) \\ &\cong \bigoplus_{i=0}^{\infty} S^i(\mathfrak{h}^*) \otimes \tau \Big/ \bigoplus_{i=1}^{\infty} S^i(\mathfrak{h}^*) \otimes \tau \\ &\cong S^0(\mathfrak{h}^*) \otimes \tau \\ &\cong \Bbbk \otimes \tau \\ &\cong \tau. \end{aligned}$$

So $L_{0,0}(\tau)$ is isomorphic to τ as a vector space. To consider how the rational Cherednik algebra $H_{0,0}(G,\mathfrak{h})$ acts on $L_{0,0}(\tau)$, let $v \in \tau$. For any $g \in G$, the vector $g.v$ is determined by the representation τ. For any $y \in \mathfrak{h}$, we have $y.v = 0$ by definition of the Verma module. Finally for any $x \in \mathfrak{h}^*$ we have $x.v = 0$, because $\text{degree}(x.v) = \text{degree}(v) + 1 = 1$ and we took a quotient by everything with positive degree. Therefore $L_{0,0}(\tau)$ is a module for the Cherednik alebra $H_{0,0}(G,\mathfrak{h})$ which behaves just like the representation τ with the usual action of G, whilst $\mathfrak{h}$ and $\mathfrak{h}^*$ act by zero.

We now consider general values for t and c once again. The next proposition shows us that singular vectors generate proper graded submodules, which shows their importance in finding the maximal proper graded submodule of a Verma module.

Proposition 2.4.3. *Suppose $f \in M$ is a singular vector in some $\mathbb{Z}$-graded $H_{t,c}(G,\mathfrak{h})$-module* *with degrees bounded from below* $M = \bigoplus_{d \geq l} M^d$*. The* $H_{t,c}(G, \mathfrak{h})$*-submodule generated by* f *is a proper graded submodule of* M*.*

Proof. Let $f \in M$ be a singular vector and consider the submodule generated by f under the action of $H_{t,c}(G, \mathfrak{h})$.
Since f is homogeneous and $H_{t,c}(G, \mathfrak{h})$ has a graded action on graded modules, the submodule generated by f is a graded submodule. This submodule is spanned by linear combinations of the elements $a.f$ for some $a \in H_{t,c}(G, \mathfrak{h})$. By the PBW theorem, these can be written as linear combinations of elements of the form

$$a.f = x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} g y_1^{b_1} \cdots y_m^{b_m}.f.$$

However, because f is singular we have $y_j.f = 0$ for all $j \in \{1, \ldots, m\}$. Therefore the only nonzero elements of the submodule generated by f are those which can be written as linear combinations of elements of the form $x_1^{a_1} \cdots x_m^{a_m} g.f$. Now

$$\mathrm{degree}(x_1^{a_1} \cdots x_m^{a_m} g.f) = a_1 + \cdots + a_m + \mathrm{degree}(f) \geq \mathrm{degree}(f) > l$$

because f is a singular vector so the degree of f is strictly larger than l. Hence the submodule contains no elements of degree l and must therefore be proper. □

Let $S(\mathfrak{h}^*)^G_+$ denote the set of G-invariant elements of strictly positive degree in $S(\mathfrak{h}^*)$.

Proposition 2.4.4. *Let the characteristic of* $\Bbbk$ *be arbitrary. For every* c*, if* $f \in S(\mathfrak{h}^*)^G_+$ *then* $f \otimes v$ *is singular in* $M_{0,c}(\tau)$ *for any* $v \in \tau$*.*

Proof. Let $f \in S(\mathfrak{h}^*)^G_+$ be a G-invariant of positive degree in $S(\mathfrak{h})^*$ and choose any $v \in \tau$. The action of $y \in \mathfrak{h}$ on $f \otimes v \in M_{0,c}(\tau)$ is given by the Dunkl operator

$$y.(f \otimes v) = -\sum_{s \in S} c_s \langle \alpha_s, y \rangle \frac{(1-s).f}{\alpha_s} \otimes s.v$$

Since f is G-invariant, we have $(1-s).f = 0$ for all $s \in S$. Therefore for every c, we have $y.(f \otimes v) = 0$ for all $y \in \mathfrak{h}$. □

Proposition 2.4.5. *Let the characteristic of* $\Bbbk$ *be a prime* p*. For every* c*, if* $f \in S(\mathfrak{h}^*)^G_+$ *then* $f^p \otimes v$ *is singular in* $M_{1,c}(\tau)$ *for any* $v \in \tau$*.*

Proof. Let $f \in S(\mathfrak{h}^*)^G_+$ be a G-invariant of positive degree in $S(\mathfrak{h}^*)$ and choose any $v \in \tau$. The action of $y \in \mathfrak{h}$ on $f^p \otimes v \in M_{1,c}(\tau)$ is given by the Dunkl operator

$$y.(f^p \otimes v) = \partial_y(f^p) \otimes v - \sum_{s \in S} c_s \langle \alpha_s, y \rangle \frac{(1-s).f^p}{\alpha_s} \otimes s.v.$$

Since f is G-invariant, we have

$$(1-s)\centerdot f^p = ((1-s)\centerdot f)^p) = 0$$

for all $s \in S$. Furthermore,

$$\partial_y(f^p) = pf^{p-1}\partial_y(f) = 0$$

in characteristic p. Therefore for every c, we have $y\centerdot(f^p \otimes v) = 0$ for all $y \in \mathfrak{h}$. □

Lemma 2.4.6. *If f is a singular vector then $g\centerdot f$ is a singular vector for all $g \in G$.*

Proof. Let $y \in \mathfrak{h}$ be arbitrary. Since f is singular, we have $y\centerdot f = 0$. For all $g \in G$ we have $g^{-1}\centerdot y \in \mathfrak{h}$, therefore

$$y\centerdot(g\centerdot f) = g\centerdot(g^{-1}\centerdot y)\centerdot f = g\centerdot 0 = 0.$$

Hence $g\centerdot f$ is a singular vector for any $g \in G$. □

Corollary 2.4.7. *The singular vectors in each graded piece of a $H_{t,c}(G,\mathfrak{h})$-module form a G subrepresentation. Hence if no graded piece of a $H_{t,c}(G,\mathfrak{h})$-module has an irreducible G subrepresentation consisting of singular vectors, then it has no singular vectors and is irreducible.*

2.5 Baby Verma modules

In characteristic 0, for almost all choices of the parameter c, the Verma module $M_{1,c}(\tau)$ is irreducible. However this never happens when $t = 0$ or in characteristic p, because invariants of the reflection group, or their p^{th} powers, are central and thus lead to large submodules. Since our goal is to obtain the simple quotient of a Verma module, it is often useful first to consider the quotient by this large submodule.

At $t = 0$, the subalgebra $S(\mathfrak{h}^*)^G_+$ is central in $H_{0,c}(G,\mathfrak{h})$ so $\left(S(\mathfrak{h})^G_+\right) M_{0,c}(\tau)$ is a proper submodule of $M_{0,c}(\tau)$. This is in fact the submodule generated by singular vectors of the form described in Proposition 2.4.4. The following definition can be found in [Go03] for characteristic 0, and in [BaCh13a] for characteristic p.

Definition 2.5.1. The *baby Verma module at* $t = 0$ is the quotient

$$N_{0,c}(\tau) = \frac{M_{0,c}(\tau)}{\left(S(\mathfrak{h}^*)^G_+\right) M_{0,c}(\tau)}$$

of the Verma module $M_{0,c}(\tau)$.

Suppose that $\Bbbk$ has positive characteristic p and consider the subalgebra $\left(S(\mathfrak{h}^*)^G_+\right)^p$ whose elements are the p^{th} powers of elements in $S(\mathfrak{h}^*)^G_+$. At $t = 1$ the subalgebra $\left(S(\mathfrak{h}^*)^G_+\right)^p$ is central in $H_{1,c}(G,\mathfrak{h})$ so $\left(S(\mathfrak{h})^G_+\right)^p M_{1,c}(\tau)$ is a proper submodule of $M_{1,c}(\tau)$. This submodule is generated by singular vectors of the form described in Proposition 2.4.5. The following definition is unique to positive characteristic and can be found in [BaCh13a].

Definition 2.5.2. The *baby Verma module at $t = 1$* is the quotient

$$N_{1,c}(\tau) = \frac{M_{1,c}(\tau)}{\big(S(\mathfrak{h}^*)_+^G\big)^p M_{1,c}(\tau)}$$

of the Verma module $M_{1,c}(\tau)$.

In 1954, Geoffrey Colin Shephard and John Arthur Todd classified the complex reflection groups and detailed the degrees of their fundamental invariants [ShTo54]. Moreover, the Chevalley–Shephard–Todd theorem (Theorem 3.3.1) states that the invariants of reflection groups generate a polynomial algebra of rank equal to the dimension of $\mathfrak{h}$. This information can be used to prove the following proposition.

Proposition 2.5.3 ([BaCh13a], Proposition 2.16)**.** *Baby Verma modules are finite-dimensional.*

Corollary 2.5.4. *The irreducible quotients $L_{t,c}(\tau)$ of Verma modules are finite-dimensional.*

Proof. A baby Verma module is a quotient of Verma module by a proper graded submodule. The quotient of the Verma module $M_{t,c}(\tau)$ by its *maximal* proper graded submodule is the irreducible module $L_{t,c}(\tau)$. Therefore the irreducible module $L_{t,c}(\tau)$ is at least as small as the baby Verma module $N_{t,c}(\tau)$. By Proposition 2.5.3, baby Verma modules are finite-dimensional, so it follows that the irreducible modules $L_{t,c}(\tau)$ are also finite-dimensional. □

Every singular vector in a Verma module generates a proper graded submodule and, in some cases, the maximal proper graded submodule $J_{t,c}(\tau)$ is generated by all these singular vectors. In such cases, the maximal proper graded submodule is the sum of all proper graded submodules of the Verma module. However this is not always the case, since it is possible that the quotient of the Verma module by the submodule generated from all its singular vectors is not irreducible. An iterative process is therefore required to calculate $J_{t,c}(\tau)$.

Let J_0 denote the proper graded submodule generated by all of the singular vectors in $M_{t,c}(\tau)$. If $M_{t,c}(\tau)/J_0$ is an irreducible module, then $J_0 = J_{t,c}(\tau)$.
Otherwise, if the quotient $M_{t,c}(\tau)/J_0$ is not irreducible then denote by J_1 the proper graded submodule generated by all singular vectors in $M_{t,c}(\tau)/J_0$. Elements of the submodule J_1 are singular because their images under all Dunkl operators fall within the submodule J_0. Since J_1 is a subset of $M_{t,c}(\tau)/J_0$, we can lift it to $M_{t,c}(\tau)$ by considering its preimage $\widetilde{J_1}$ under the quotient map $M_{t,c}(\tau) \twoheadrightarrow M_{t,c}(\tau)/J_0$. Thus $\left(M_{t,c}(\tau)/J_0\right)/J_1 \cong M_{t,c}(\tau)/\widetilde{J_1}$. If $M_{t,c}(\tau)/\widetilde{J_1}$ is irreducible, then $\widetilde{J_1} = J_{t,c}(\tau)$. Otherwise, $M_{t,c}(\tau)/\widetilde{J_1}$ must still contain singular vectors which contribute to the maximal proper graded submodule. For an example of this happening, see Theorem 12.1.3 where we calculate a vector which is only singular modulo the quotient by another singular vector of lower degree.

This process will always terminate in finitely many steps because by first taking a quotient to the baby Verma module, we already have something finite-dimensional. Since the dimension of a module decreases each time we take a proper quotient, there will only be finitely many more times this procedure can continue before obtaining the irreducible module $L_{t,c}(\tau)$.

2.6 Category $\mathcal{O}$

Category $\mathcal{O}$ is a concept which first arose in the representation theory of semisimple complex Lie algebras. Since its inception the idea has been applied in various other contexts. The definition given here is quite different than we see elsewhere, but it is chosen to replicate the same behaviour as any other category $\mathcal{O}$.

Definition 2.6.1 ([BaCh13a])**.** Category $\mathcal{O}_{t,c}(G,\mathfrak{h})$ in characteristic p is a category whose objects are $\mathbb{Z}$-graded finite-dimensional representations of the rational Cherednik algebra $H_{t,c}(G,\mathfrak{h})$ over $\Bbbk$, and the morphisms are homomorphisms of representations that preserve grading up to uniform shift by a constant.

We will abbreviate notation as $\mathcal{O}_{t,c}$ when the algebra is clear from context. Unlike other categories $\mathcal{O}$, Verma modules are not objects in the category $\mathcal{O}_{t,c}$ because they are infinite-dimensional. However, the baby Verma modules do belong in category $\mathcal{O}_{t,c}$. The following two theorems illustrate that this definition of category $\mathcal{O}_{t,c}$ is well chosen.

Theorem 2.6.2 ([BaCh13a])**.** *The irreducible quotient $L_{t,c}(\tau)$ of the Verma module $M_{t,c}(\tau)$ is in category $\mathcal{O}_{t,c}(G,\mathfrak{h})$.*

Proof. Verma modules are $\mathbb{Z}$-graded and this grading descends to quotients by graded submodules. Furthermore, we have seen in Corollary 2.5.4 that the irreducible modules $L_{t,c}(\tau)$ are finite-dimensional. □

Theorem 2.6.3 ([BaCh13a])**.** *Every simple object in the category $\mathcal{O}_{t,c}(G,\mathfrak{h})$ is isomorphic to the unique irreducible quotient $L_{t,c}(\tau)$ of a Verma module $M_{t,c}(\tau)$ for some irreducible representation $\tau \in \widehat{G}$,*

The set $\widehat{G}$ of irreducible representations of G is in one-to-one correspondence with Verma modules. Every simple object in $\mathcal{O}_{t,c}$ is the unique simple quotient of some Verma module and no two are isomorphic. Consequently, the simple objects of $\mathcal{O}_{t,c}$ are also in one-to-one correspondence with irreducible representations $\tau \in \widehat{G}$. We therefore have a strategy for calculating all the simple objects in $\mathcal{O}_{t,c}$. We begin with some irreducible representation of the reflection group, construct its corresponding Verma module, and then quotient by the unique maximal proper graded submodule. By repeating this procedure for each irreducible representation of the group, we obtain all the simple objects in $\mathcal{O}_{t,c}$. However, this does not tell us about extensions or blocks in the category which is a problem often in consideration. For more detail, see the literature survey in Chapter 5.

Our goal is to describe every simple object in $\mathcal{O}_{t,c}$ which can be done by giving their characters or Hilbert polynomials, which we now define.

Let $\mathrm{Rep}(G)$ be the category of finite-dimensional representations of G and let $K_0\big(\mathrm{Rep}(G)\big)$ be its Grothendieck group. Let $M = \bigoplus_{i\in\mathbb{Z}} M^i$ be a $\mathbb{Z}$-graded $H_{t,c}(G,\mathfrak{h})$-module with finite-dimensional graded pieces M^i.

Definition 2.6.4. For any $\mathbb{Z}$-graded module M we denote by $M[k]$ the same module but with the grading shifted by k. That is, $M^i[k] = M^{i+k}$.

Since G has a degree-preserving action on the $H_{t,c}(G,\mathfrak{h})$-modules, we can consider each graded piece M^i as a finite-dimensional representation of G. We use the notation $[M^i]$ to represent the isomorphism class of M^i in the Grothendieck group $K_0\big(\mathrm{Rep}(G)\big)$.

Definition 2.6.5. The *character* of M is the power series

$$\chi_M(z) = \sum_i [M^i] z^i$$

in formal variables z, z^{-1} with coefficients in $K_0\big(\mathrm{Rep}(G)\big)$.

In other words, the character records how each graded piece M^i of the graded module M decomposes into finite-dimensional representations of G.

Example 2.6.6. Let τ be an irreducible representation of G and consider the Verma module $M_{t,c}(\tau)$. The character of $M_{t,c}(\tau)$ is

$$\chi_{M_{t,c}(\tau)}(z) = \sum_{i=0}^{\infty} [S^i(\mathfrak{h}^*) \otimes \tau] z^i.$$

Related to the character of a graded module is the idea of its Hilbert series.

Definition 2.6.7. The *Hilbert series* of M is the power series

$$\mathrm{Hilb}_M(z) = \sum_i (\dim\, M^i)\, z^i$$

in formal variables z, z^{-1}.

In a Hilbert series, the coefficient of z^i records the dimension of M^i as a finite-dimensional vector space. If a Hilbert series has only finitely many nonzero terms and all the exponents of z are non-negative then the Hilbert series is called the *Hilbert polynomial*.

Example 2.6.8. The Hilbert series of the Verma module $M_{t,c}(\tau) \cong S(\mathfrak{h}^*) \otimes \tau$ is

$$\mathrm{Hilb}_{M_{t,c}(\tau)}(z) = \frac{\dim\,\tau}{(1-z)^{\dim\,\mathfrak{h}}}\ .$$

For the next two examples, suppose that the fundamental invariants which generate the algebra $(S(\mathfrak{h}^*))^G$ have degrees $d_1, \ldots, d_m$ where $m = \dim\,\mathfrak{h}$.

Example 2.6.9. The Hilbert polynomial of the baby Verma module $N_{0,c}(\tau)$ is

$$\mathrm{Hilb}_{N_{0,c}(\tau)}(z) = (1-z^{d_1})\cdots(1-z^{d_m})\frac{\dim\,\tau}{(1-z)^m}\ .$$

Example 2.6.10. The Hilbert polynomial of the baby Verma module $N_{1,c}(\tau)$ is

$$\mathrm{Hilb}_{N_{1,c}(\tau)}(z) = (1-z^{pd_1})\cdots(1-z^{pd_m})\frac{\dim\,\tau}{(1-z)^m}\ .$$

We now state Example 2.4.2 as a proposition using the language of characters and Hilbert polynomials.

Proposition 2.6.11. *Let $\Bbbk$ be an algebraically closed field of characteristic p, let the values of the parameters be $t = 0$ and $c = 0$, and let τ be any irreducible representation of G. The irreducible representation $L_{0,0}(\tau)$ of $H_{0,0}(G, \mathfrak{h})$ is the quotient of the Verma module $M_{0,0}(\tau)$ by all the positively graded vectors, with the character*

$$\chi_{L_{0,0}(\tau)}(z) = [\tau]$$

and the Hilbert polynomial

$$\mathit{Hilb}_{L_{0,0}(\tau)}(z) = \dim \tau.$$

Proof. When $t = c = 0$, the Dunkl operators are all identically equal to 0, so all vectors of strictly positive degree are singular. Thus, $L_{0,0}(\tau)$ is concentrated in degree 0, where it equals τ. □

Definition 2.6.12. Let $S^{(p)}(\mathfrak{h}^*)$ be defined as the quotient of $S(\mathfrak{h}^*)$ by the ideal generated by $\{x^p \mid x \in \mathfrak{h}^*\}$.

Proposition 2.6.13. *Let $\Bbbk$ be an algebraically closed field of characteristic p, let the values of the parameters be $t = 1$ and $c = 0$, and let τ be any irreducible representation of G. The irreducible representation $L_{1,0}(\tau)$ of $H_{1,0}(G, \mathfrak{h})$ is the quotient of the Verma module $M_{1,0}(\tau)$ by all the vectors of the form $x^p \otimes v$ for any $x \in \mathfrak{h}^*$ and $v \in \tau$. It has the character*

$$\chi_{L_{1,0}(\tau))}(z) = \chi_{S^{(p)}(\mathfrak{h}^*)}(z) \cdot [\tau]$$

and the Hilbert polynomial

$$Hilb_{L_{1,0}(\tau)}(z) = \left(\frac{1-z^p}{1-z}\right)^{\dim \mathfrak{h}} \dim \tau.$$

Proof. This is well known, see also Theorem 3.1 in [Li14]. When $t = 1$ and $c = 0$, the Dunkl operators have a very simple form $D_y = \partial_y \otimes \mathrm{id}$. The joint kernel of such operators for all $y \in \mathfrak{h}$ consists of vectors of the form $x^p \otimes v$ for any $x \in \mathfrak{h}^*$ and $v \in \tau$. □

2.7 Generic parameters

The parameter $c : S \to \Bbbk$ is a function comprising a choice of constants $c_1, \ldots, c_k \in \Bbbk$, with one constant for each conjugacy class of reflections. Since this parameter c appears in the definition of a Dunkl operator, singular vectors and their behaviour depend therefore on the choice of c. When one choice of this parameter gives rise to the same behaviour of singular vectors as infinitely many other parameters, then this choice for the parameter is described as generic. Generic behaviour happens outside of countably many (in positive characteristic, finitely many) values [BEG03a].

Since singular vectors are homogeneous elements of graded modules, we can compute them by considering each graded degree in turn. In each degree, we can pick a basis for the graded module so that the Dunkl operators $D_{y_1}, \ldots, D_{y_m}$ acting in that degree may be written as matrices. Taking the kernels of these matrices and considering their intersection, we obtain all the singular vectors in that degree of the module.

The ranks of these matrices depend polynomially on $c_1, \ldots, c_k$. To see this, consider the matrices in row echelon form; the entries along the leading diagonal consist of polynomials in $c_1, \ldots, c_k$. Those polynomials and their roots determine a finite union of hyperplanes in the function space whose points are different choices of the parameter c. Every value of c chosen outside this finite union of hyperplanes leaves the ranks of the matrices unchanged, and such a choice of c is called *generic*. For a fixed irreducible representation $\tau \in \widehat{G}$, the simple $H_{t,c}(G, \mathfrak{h})$-module $L_{t,c}(\tau)$ at every generic value of c has the same character and Hilbert polynomial.

When c takes its value inside the finite union of hyperplanes described above, then some of the values among $c_1, \ldots, c_k$ are roots of polynomials along the leading diagonal of some row echelon matrix. In this case, the rank of some matrices will be reduced. If for each $\tau \in \widehat{G}$, the character and Hilbert series of the simple module $L_{t,c}(\tau)$ at this value of c is equal to the character and Hilbert series of $L_{t,c}(\tau)$ at generic values of c, then this value of c is *also* generic as the behaviour is unchanged.

However, by reducing the ranks of the matrices, it is possible that this choice of c will increase the intersection of their kernels and produce additional singular vectors. In such cases, the parameter c is called *non-generic* and the simple module $L_{t,c}(\tau)$ will have a different character and a strictly smaller Hilbert series than the simple module at generic values of c. We also refer to non-generic values of c as *special*.

2.8 Twist by a character

In this section we describe a correspondence which occurs between the simple objects in the categories $\mathcal{O}$ for two rational Cherednik algebras which are related by a twist. Simple objects in a category $\mathcal{O}$ are parametrised by irreducible representations. When different irreducible representations are related through the tensor product with a linear character, we can learn about one simple object by considering instead the corresponding simple object for the twisted algebra.

Let χ be any linear character (1-dimensional representation) of G over $\Bbbk$ with its basis vector labelled 1_χ. Given $g \in G$, the linear map $\chi(g)$ acts on 1_χ by a constant and we identify $\chi(g)$ with this constant.
Consider the rational Cherednik algebra $H_{t,\chi c}(G, \mathfrak{h})$ with $\chi c : S \to \Bbbk$ defined by $s \mapsto \chi(s)c(s)$.

Proposition 2.8.1. *The map $\Phi : H_{t,c}(G, \mathfrak{h}) \to H_{t,\chi c}(G, \mathfrak{h})$ defined by*

$$\Phi(x) = x, \quad \Phi(y) = y, \quad \Phi(g) = \chi(g)g$$

for all $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, and $g \in G$, is a graded isomorphism of rational Cherednik algebras.

Proof. We will show that the map Φ defines an injective and surjective homomorphism of algebras.
To show that Φ defines a homomorphism, let us check that the defining relations in the algebras are preserved under this map. First, Φ defines an endomorphism of $\Bbbk[G] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$. The relations which define $H_{t,c}(G, \mathfrak{h})$ as a quotient of $\Bbbk[G] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ are

$$\begin{aligned} 0 &= xx' - x'x, \\ 0 &= yy' - y'y, \\ 0 &= xy - yx - \langle x, y\rangle t + \sum_{s \in S} \langle \alpha_s, y\rangle \langle x, \alpha_s^\vee \rangle c(s)s, \end{aligned}$$

and the relations defining $H_{t,\chi c}(G, \mathfrak{h})$ are

$$\begin{aligned} 0 &= xx' - x'x, \\ 0 &= yy' - y'y, \\ 0 &= xy - yx - \langle x, y\rangle t + \sum_{s \in S} \langle \alpha_s, y\rangle \langle x, \alpha_s^\vee \rangle \chi(s)c(s)s \end{aligned}$$

for all $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$. We have

$$\Phi(xx' - x'x) = \Phi(x)\Phi(x') - \Phi(x')\Phi(x) = xx' - x'x = 0$$

therefore the map Φ preserves this relation. By this same argument, the second relation is similarly preserved by Φ. Finally,

$$\begin{aligned}
&\Phi\Big(xy - yx - \langle x, y\rangle t + \sum_{s\in S}\langle \alpha_s, y\rangle\langle x, \alpha_s^\vee\rangle c(s)s\Big) \\
&= \Phi(x)\Phi(y) - \Phi(y)\Phi(x) - \langle x, y\rangle t + \sum_{s\in S}\langle \alpha_s, y\rangle\langle x, \alpha_s^\vee\rangle c(s)\Phi(s) \\
&= xy - yx - \langle x, y\rangle t + \sum_{s\in S}\langle \alpha_s, y\rangle\langle x, \alpha_s^\vee\rangle c(s)\chi(s)s \\
&= 0
\end{aligned}$$

which shows the third defining relation is preserved.

Injectivity and surjectivity are easy to see, because Φ maps a basis of the domain to a basis of the codomain. We can even construct an inverse map Φ^{-1} with $\Phi^{-1}(x) = x$, $\Phi^{-1}(y) = y$ and $\Phi^{-1}(g) = \chi^{-1}(g)g$.

This shows that Φ is an injective, surjective homomorphism of rational Cherednik algebras, therefore $H_{t,c}(G, \mathfrak{h})$ is isomorphic to $H_{t,\chi c}(G, \mathfrak{h})$. □

Since Φ defines an isomorphism from $H_{t,c}(G, \mathfrak{h})$ to $H_{t,\chi c}(G, \mathfrak{h})$, given any $H_{t,\chi c}(G, \mathfrak{h})$-module M, there is a $H_{t,c}(G, \mathfrak{h})$-module $\Phi^*(M)$ called the *pullback along Φ*.

Proposition 2.8.2. *Let Φ be defined as in Proposition 2.8.1. The pullback module $\Phi^*\left(L_{t,\chi c}(\tau)\right)$ is isomorphic to $L_{t,c}(\chi \otimes \tau)$.*

Proof. Let $M_{t,\chi c}(\tau)$ be a Verma module for $H_{t,\chi c}(G, \mathfrak{h})$. We will first show that the pullback module $\Phi^*(M_{t,\chi c}(\tau))$ is isomorphic to $M_{t,c}(\chi \otimes \tau)$ as a representation of $H_{t,c}(G, \mathfrak{h})$. Considering only their underlying structure as vector spaces, we have

$$\Phi^*\big(M_{t,\chi c}(\tau)\big) = M_{t,\chi c}(\tau) \cong S\mathfrak{h}^* \otimes \tau$$

and

$$M_{t,c}(\chi \otimes \tau) \cong S(\mathfrak{h}^*) \otimes (\chi \otimes \tau).$$

The linear map $\tau \to \chi \otimes \tau$ defined by $v \mapsto 1_\chi \otimes v$ is an isomorphism of vector spaces, so it follows that

$$S(\mathfrak{h}^*) \otimes \tau \cong S(\mathfrak{h}^*) \otimes (\chi \otimes \tau)$$

as vector spaces. Therefore we have an isomorphism of vector spaces

$$\Psi : \Phi^*\big(M_{t,\chi c}(\tau)\big) \xrightarrow{\sim} M_{t,c}(\chi \otimes \tau)$$

given by

$$\Psi(f \otimes v) = f \otimes 1_\chi \otimes v.$$

We will prove that Ψ is an isomorphism of $H_{t,c}(G,\mathfrak{h})$-modules by showing that the map commutes with the action of the algebra,

$$\Psi\big(a_\bigstar(f \otimes v)\big) = a.\Psi(f \otimes v)$$

for all $a \in H_{t,c}(G,\mathfrak{h})$ and all $f \otimes v \in S(\mathfrak{h}^*) \otimes \tau$. In this equation, the action of $a \in H_{t,c}(G,\mathfrak{h})$ on the left hand side is defined through the pullback along Φ,

$$a_\bigstar(f \otimes v) = \Phi(a).(f \otimes v).$$

It is enough to show that the action commutes for $a = x$, $a = y$, and $a = g$ for any $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $g \in G$ because such elements generate $H_{t,c}(G,\mathfrak{h})$. Considering the action of x,

$$\begin{aligned}
\Psi\big(x_\bigstar(f \otimes v)\big) &= \Psi\big(\Phi(x).(f \otimes v)\big) \\
&= \Psi\big(x.(f \otimes v)\big) \\
&= \Psi\big((xf) \otimes v\big) \\
&= (xf) \otimes 1_\chi \otimes v \\
&= x.(f \otimes 1_\chi \otimes v) = x.\Psi\big(f \otimes v\big)
\end{aligned}$$

which shows the action of x commutes with Ψ. Similarly,

$$\begin{aligned}
\Psi\big(y_\bigstar(f \otimes v)\big) &= \Psi\big(\Phi(y).(f \otimes v)\big) \\
&= \Psi\big(y.(f \otimes v)\big) \\
&= \Psi\Big(\partial_y(f) \otimes v - \sum_{s \in S} \chi(s)c(s)\langle \alpha_s, y\rangle \frac{(1-s).f}{\alpha_s} \otimes s.v\Big) \\
&= \partial_y(f) \otimes 1_\chi \otimes v - \sum_{s \in S} \chi(s)c(s)\langle \alpha_s, y\rangle \frac{(1-s).f}{\alpha_s} \otimes 1_\chi \otimes s.v \\
&= \partial_y(f) \otimes 1_\chi \otimes v - \sum_{s \in S} c(s)\langle \alpha_s, y\rangle \frac{(1-s).f}{\alpha_s} \otimes s.1_\chi \otimes s.v \\
&= \partial_y(f) \otimes (1_\chi \otimes v) - \sum_{s \in S} c(s)\langle \alpha_s, y\rangle \frac{(1-s).f}{\alpha_s} \otimes \big(s.(1_\chi \otimes v)\big) \\
&= y.\big(f \otimes (1_\chi \otimes v)\big) \\
&= y.\Psi\big(f \otimes v\big).
\end{aligned}$$

Finally,

$$\Psi\big(g_\bigstar(f \otimes v)\big) = \Psi\big(\Phi(g).(f \otimes v)\big)$$

$$
\begin{aligned}
&= \Psi\big(\chi(g)g.(f \otimes v)\big)\\
&= \Psi\big(\chi(g)(g.f) \otimes (g.v)\big)\\
&= \chi(g)(g.f) \otimes 1_\chi \otimes (g.v)\\
&= g.f \otimes \chi(g)1_\chi \otimes g.v\\
&= g.f \otimes g.1_\chi \otimes g.v\\
&= g.(f \otimes 1_\chi \otimes v) = g.\Psi(f \otimes v).
\end{aligned}
$$

Hence the map Ψ commutes with the action of each $a \in H_{t,c}(G, \mathfrak{h})$ where a is a generator of the algebra. Therefore Ψ an isomorphism of $H_{t,c}(G, \mathfrak{h})$-modules, and the pullback module $\Phi^*(M_{t,\chi c}(\tau))$ is isomorphic to $M_{t,c}(\chi \otimes \tau)$. This gives a correspondence between the Verma module $M_{t,\chi c}(\tau)$ in the category $\mathcal{O}_{t,\chi c}(G, \mathfrak{h})$ and the Verma module $M_{t,c}(\chi \otimes \tau)$ in the category $\mathcal{O}_{t,c}(G, \mathfrak{h})$. The irreducible quotients by the maximal proper graded submodules, $L_{t,c}(\chi \otimes \tau)$ and $\Phi^*(L_{t,\chi c}(\tau))$, are consequently isomorphic. □

Corollary 2.8.3. *The characters of $L_{t,c}(\chi \otimes \tau)$ and $L_{t,\chi c}(\tau)$ are related by the formula*

$$\chi_{L_{t,c}(\chi \otimes \tau)}(z) = \chi_{L_{t,\chi c}(\tau)}(z) \cdot [\chi].$$

Corollary 2.8.4. *The Hilbert series of $L_{t,c}(\chi \otimes \tau)$ and $L_{t,\chi c}(\tau)$ are related by the formula*

$$Hilb_{L_{t,c}(\chi \otimes \tau)}(z) = Hilb_{L_{t,\chi c}(\tau)}(z).$$

Chapter 3

The Symmetric Group and Its Invariants

In this chapter we will consider the symmetric group and its reflection representations. The symmetric group is a reflection group but in order to construct its rational Cherednik algebra we must specify which reflection representation we are using. We will also describe the theory of symmetric polynomials, because an understanding of the reflection group invariants is required in order to construct baby Verma modules.

3.1 The permutation representation

Let n be a natural number. We denote by S_n the symmetric group with $n!$ elements which acts on the set $\{1, \ldots, n\}$. The transposition swapping i and j is written (i, j) or (ij).
Let V be the permutation representation of S_n with basis $\{y_1, \ldots, y_n\}$ over $\Bbbk$ on which S_n acts by permuting the basis. The element $Y = y_1 + \cdots + y_n \in V$ is a basis for the subrepresentation

$$\Bbbk Y = \operatorname{span}_{\Bbbk}\{y_1 + \cdots + y_n\} \subset V$$

of V on which S_n acts trivially, thus $\Bbbk Y \cong \texttt{triv} \in \widehat{S_n}$. The subrepresentation

$$\mathfrak{h} = \left\{ \sum_{i=1}^{n} a_i y_i \in V \;\middle|\; \sum_{i=1}^{n} a_i = 0 \right\} \subset V$$

of V has a basis given by $\{\, y_1 - y_n,\, y_2 - y_n,\, \ldots,\, y_{n-1} - y_n \,\}$ and $\{\, y_i - y_j \;\mid\; i \neq j \,\}$ is a spanning set. This subrepresentation is also known as the *standard* representation which we shall usually denote by $\texttt{stand}$. The only exception is when $n = 2$, when we have $\mathfrak{h} \cong \texttt{sign}$.

The permutation representation V is a reflection representation of S_n with the set of reflections $S = \{(ij) \mid 1 \leq i < j \leq n\}$ comprising all transpositions. For all $(n, p) \neq (2, 2)$ the standard representation $\mathfrak{h}$ of S_n over a field of characteristic p is also a reflection representation with the same generating set S of reflections. When $(n,p) = (2,2)$, there are no reflections in S_2 because $\mathfrak{h} \cong \texttt{triv}$ in this case, so the group action is not faithful and S_2 fixes all of $\mathfrak{h}$ rather than any hyperplanes of $\mathfrak{h}$. In Proposition 1.1.7, we showed that V (respectively $\mathfrak{h}$) is a reflection representation of S_n if and only if V^* (respectively $\mathfrak{h}^*$) is also a reflection representation of S_n. We explore the structure of these dual representations in Section 3.2.

Lemma 3.1.1. *V has exactly two proper subrepresentations, $\Bbbk Y$ and $\mathfrak{h}$.*

Proof. Suppose $W \leq V$ is a subrepresentation, and $W \neq 0$. For any nonzero $w \in W$ we write $w = \sum_{i=1}^{n} a_i y_i$ for some coefficients $a_i \in \Bbbk$ not all zero. Let l be the number of coefficients a_i which are nonzero and choose a nonzero $w \in W$ with the minimum value of l.

Case 1 $l = 1$.

We write $w = a_i y_i$ for some $i \in \{1, \ldots, n\}$ and $a_i \in \Bbbk$ nonzero. Therefore $y_i \in W$ and $(ij).y_i = y_j \in W$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$. Hence $W = V$ because W contains a basis for V.

Case 2 $l = 2$.

We write $w = a_i y_i + a_j y_j$ for some $i, j \in \{1, \ldots, n\}$ with $a_i, a_j \neq 0$ and $i \neq j$. Now $(ij).w = a_j y_i + a_i y_j \in W$. We can multiply this by the nonzero scalar $\frac{a_j}{a_i}$ to get $\frac{a_j^2}{a_i} y_i + a_j y_j \in W$. Subtracting this from w gives $a_i y_i - \frac{a_j^2}{a_i} y_i = \left(\frac{a_i^2 - a_j^2}{a_i}\right) y_i \in W$. By assumption every nonzero element of W has at least 2 nonzero coefficients. Therefore since $\left(\frac{a_i^2 - a_j^2}{a_i}\right) y_i$ is an element of W with only 1 coefficient, it must be zero. Hence $\frac{a_i^2 - a_j^2}{a_i} = 0$ which implies $a_i^2 = a_j^2$.

Case 2.1 $a_j = -a_i$. Therefore $\frac{1}{a_i} w = y_i - y_j \in W$. Through the action of S_n on this element of W we can obtain every element in the basis of $\mathfrak{h}$, hence $\mathfrak{h} \leq W$. The dimension of $\mathfrak{h}$ is $n-1$ therefore W has dimension $n-1$ or n; however W cannot have dimension n because then $W = V$ which implies W contains elements with $l < 2$. Hence $W = \mathfrak{h}$.

Case 2.2 $a_j = a_i$. Therefore $\frac{1}{a_i} w = y_i + y_j \in W$.

Suppose $n = 2$, so $w = y_1 + y_2$ and $\operatorname{span}_{\Bbbk}\{y_1 + y_2\} = \Bbbk Y \leq W$. The dimension of $\Bbbk Y$ is 1, and W has greater or equal dimension so it is dimension 1 or 2. However, if the dimension of W is 2 then $W = V$ which is a contradiction because V contains elements with $l < 2$. Hence $W = \Bbbk Y$ when $n = 2$.

If $n > 2$ then choose $k \in \{1, \ldots n\}$ with $k \neq i, j$. Now $(ik).(y_i + y_j) = y_k + y_j \in W$. Subtracting w gives $y_k - y_i \in W$, and $(ijk).(y_k - y_i) = y_i - y_j \in W$. Therefore $(y_i + y_j) - (y_i - y_j) = 2y_j \in W$. If the characteristic of $\Bbbk$ is 2 then Case 2.2 is equivalent to Case 2.1 so $W = \mathfrak{h}$ and we are done. However if the characteristic is not 2, then $y_j \in W$ is nonzero and this is a contradiction because we have found an element of W with $l < 2$.

Case 3 $l \geq 3$.

We can write $w = a_1y_1 + \cdots + a_ly_l$ where $a_1, \ldots, a_l \neq 0$ using the action of S_n to reorder the coefficients without loss of generality.
Suppose $\exists\, i, j \in \{1, \ldots, l\}$ with $a_i \neq a_j$. Without loss of generality, $j = l$ and $i = l-1$ so $a_{l-1} \neq a_l$. Now

$$w - \frac{a_l}{a_{l-1}} (l-1,\ l) \,.\, w = \sum_{k=1}^{l-2} \left(a_k - \frac{a_l}{a_{l-1}} a_k \right) y_k + \left(\frac{a_{l-1}^2 - a_l^2}{a_{l-1}} \right) y_{l-1} \in W.$$

This expression has fewer coefficients than w and must therefore equal 0, however

$$\left(a_k - \frac{a_l}{a_{l-1}} a_k \right) = a_k \left(1 - \frac{a_l}{a_{l-1}} \right) = 0 \implies a_k \big(a_{l-1} - a_l \big) = 0$$

and this is a contradiction as both $a_k \neq 0$ and $(a_{l-1} - a_l) \neq 0$ for all $k \in \{1 \ldots, l-2\}$.
Therefore we do not have any $i, j \in \{1, \ldots, l\}$ with $a_i \neq a_j$, so $a_1 = a_2 = \cdots = a_l$.
Hence $\frac{1}{a_1} w = y_1 + \cdots + y_l \in W$. If $l = n$, then $\operatorname{span}_{\Bbbk}\{y_1 + \cdots + y_n\} \leq W$. By assumption no element of W can be written with fewer than n nonzero coefficients so $W = \Bbbk Y$.
Suppose $l < n$, so $y_1 + \cdots + y_l \in W$. We have

$$\big(y_1 + \cdots + y_l\big) - (l,\ l+1) . \big(y_1 + \cdots + y_l\big) = y_l - y_{l+1} \in W$$

which is an element of W with only 2 nonzero coefficients, which contradicts the choice of w. □

The permutation representation V is an n-dimensional representation and contains subrepresentations of dimensions 1 and $(n-1)$. The following lemma tells us when V decomposes as a direct sum of these two subrepresentations.

Lemma 3.1.2. *There is a direct sum decomposition $V = \mathfrak{h} \oplus \Bbbk Y$ if and only if $p \nmid n$. Moreover, if $p \mid n$ then $\Bbbk Y \subseteq \mathfrak{h}$.*

Proof. If $p \mid n$, then the nonzero vector $Y = y_1 + \cdots + y_n$ has coefficients which sum to zero, and therefore Y lies in the intersection $\mathfrak{h} \cap \Bbbk Y$ so the sum is not direct. Furthermore, any multiple of Y also has coefficients which sum to zero, therefore $\Bbbk Y \subseteq \mathfrak{h}$.
Suppose $p \nmid n$, we claim $\mathfrak{h} \cap \Bbbk Y = \{0\}$. Let $v \in (\mathfrak{h} \cap \Bbbk Y)$ and suppose for the sake of contradiction that $v = \sum_{i=1}^{n} a_i y_i$ is nonzero. Therefore $v \in \Bbbk Y$ so we have

$$v = \lambda Y = \lambda y_1 + \cdots + \lambda y_n$$

for some nonzero $\lambda \in \Bbbk$. However since $v \in \mathfrak{h}$ we have $\sum_{i=1}^{n} a_i = 0 = n\lambda$ and $p \nmid n$ so we can multiply by n^{-1} to get $\lambda = 0$ which is a contradiction. Hence the sum is direct, and $\mathfrak{h} \oplus \Bbbk Y$

is a subspace of V with dimension

$$\dim \mathfrak{h} + \dim \Bbbk Y = n - 1 + 1 = n = \dim V$$

so $\mathfrak{h} \oplus \Bbbk Y = V$. □

Lemma 3.1.3. *When $n = 2$ the standard representation $\mathfrak{h}$ is always irreducible.*

Proof. The dimension of $\mathfrak{h}$ is

$$n - 1 = 2 - 1 = 1$$

and therefore $\mathfrak{h}$ is an irreducible representation of S_2 in any characteristic.
In particular, when $p = 2$ we have $\mathfrak{h} \cong \texttt{triv}$ and when $p \geq 3$ we have $\mathfrak{h} \cong \texttt{sign}$. □

Lemma 3.1.4. *Suppose $n \geq 3$. The standard representation $\mathfrak{h}$ is irreducible if and only if $p \nmid n$.*

Proof. If $p \mid n$, then $Y \in \mathfrak{h}$ therefore $\Bbbk Y \subseteq \mathfrak{h}$ and this inclusion is proper therefore $\mathfrak{h}$ is not irreducible.
Suppose $p \nmid n$. Any proper subrepresentation of $\mathfrak{h}$ would also be a proper subrepresentation of V. By Lemma 3.1.1 the only proper subrepresentations of V are $\Bbbk Y$ and $\mathfrak{h}$, but Lemma 3.1.2 tells us that $Y \notin \mathfrak{h}$ when $p \nmid n$, therefore $\mathfrak{h}$ has no proper subrepresentations and is irreducible. □

If $p \nmid n$ then $\mathfrak{h}$ is an irreducible subrepresentation of V with a direct complement. We will now explore the structure of V in arbitrary characteristic by considering a short exact sequence. Recall that triv has a basis given by $\{1_{\texttt{triv}}\}$ with trivial action $g.1_{\texttt{triv}} = 1_{\texttt{triv}}$ for all $g \in S_n$.

Proposition 3.1.5. *There exists maps $\alpha : \mathfrak{h} \to V$ and $\beta : V \to \texttt{triv}$ such that the sequence*

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\alpha} V \xrightarrow{\beta} \texttt{triv} \longrightarrow 0$$

is a short exact sequence of representations of S_n. Therefore the quotient $V/\mathfrak{h}$ is isomorphic to a trivial representation triv.

Proof. We defined $\mathfrak{h}$ as a subrepresentation of V, so let the map α be the inclusion $\mathfrak{h} \hookrightarrow V$. This ensures the sequence is exact at $\mathfrak{h}$. We will deduce the map β by considering the condition $\text{im } \alpha = \ker \beta$.

For each basis vector $y_i \in V$ we have $\beta(y_i) = \lambda_i \cdot 1_{\texttt{triv}}$ for some $\lambda_i \in \Bbbk$. However, if $\text{im } \alpha = \ker \beta$ then everything in $\mathfrak{h}$ is sent to 0 by β because $\text{im } \alpha = \mathfrak{h}$. Let $y_i - y_j$ be an arbitrary basis vector in $\mathfrak{h}$, so

$$0 = \beta(y_i - y_j) = \beta(y_i) - \beta(y_j) = \lambda_i \cdot 1_{\texttt{triv}} - \lambda_j \cdot 1_{\texttt{triv}} = (\lambda_i - \lambda_j) \cdot 1_{\texttt{triv}} \implies \lambda_i = \lambda_j$$

which holds for all $j \neq i$. Without loss of generality we can assume $\lambda_i = 1$ for all i since any nonzero multiple of β will also give a short exact sequence. Hence $\beta(y_i) = 1_{\mathtt{triv}}$ for every basis vector y_i. We will now verify that this definition of β indeed gives us a short exact sequence.

For the sequence to be short and exact we require im $\alpha =$ ker β. We can show this, since $\mathfrak{h} = \text{span}_{\Bbbk}\{y_i - y_j \mid i \neq j\}$ and $y_i - y_j \in \ker \beta$ for all $i \neq j$, so we know that im $\alpha \subseteq$ ker β. Furthermore, $\dim V = n$ and since β is surjective we have $\dim \operatorname{im} \beta = \dim \mathtt{triv} = 1$, so $\dim \ker \beta = n - 1$. As $\dim \operatorname{im} \alpha = n - 1$ we can conclude im $\alpha =$ ker β, therefore the sequence is short and exact. □

By Lemma 3.1.2, this sequence splits if and only if $p \nmid n$ and we will now show that this condition may be deduced from the definition of a split exact sequence.

Proposition 3.1.6. *Consider the short exact sequence*

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\alpha} V \xrightarrow{\beta} \mathtt{triv} \longrightarrow 0$$

from Proposition 3.1.5, where $\alpha : \mathfrak{h} \hookrightarrow V$ is the inclusion of $\mathfrak{h}$ into V, and $\beta : V \to \mathtt{triv}$ is defined by $\beta(y_i) = 1_{\mathtt{triv}}$ for all $i \in \{1, \dots, n\}$. This sequence splits if and only if $p \nmid n$.

Proof. If the sequence splits then it is right-split, so there exists a section of β, which is a map $\delta : \mathtt{triv} \to V$ with the property that $\beta \circ \delta$ is an identity map on $\mathtt{triv}$.

Suppose the sequence splits. The section δ is determined by where it sends the basis $\{1_{\mathtt{triv}}\}$ of $\mathtt{triv}$. The image $\delta(1_{\mathtt{triv}})$ is an element of V therefore $\delta(1_{\mathtt{triv}}) = a_1 y_1 + \cdots + a_n y_n$ for some coefficients $a_1, \dots, a_n \in \Bbbk$. However, δ is a map of representations and therefore commutes with the action of S_n. For all permutations $g \in S_n$ we have

$$\begin{aligned} 1_{\mathtt{triv}} = g.1_{\mathtt{triv}} &\implies \delta(1_{\mathtt{triv}}) = \delta(g.1_{\mathtt{triv}}) \\ &\implies \delta(1_{\mathtt{triv}}) = g.\delta(1_{\mathtt{triv}}) \\ &\implies a_1 y_1 + \cdots + a_n y_n = g.(a_1 y_1 + \cdots + a_n y_n). \end{aligned}$$

The effect of g on the element $a_1 y_1 + \cdots + a_n y_n$ is to permute the coefficients. Since the result is equality for all permutations $g \in S_n$ we can conclude that all the coefficients $a_1, \dots, a_n \in \Bbbk$ are all equal. Hence there exists a nonzero constant $a \in \Bbbk$ with $\delta(1_{\mathtt{triv}}) = a(y_1 + \cdots + y_n)$. We require $\beta \circ \delta(1_{\mathtt{triv}}) = 1_{\mathtt{triv}}$, so

$$\begin{aligned} 1_{\mathtt{triv}} = \beta(\delta(1_{\mathtt{triv}})) &= \beta(a y_1 + \cdots + a y_n) \\ &= a\beta(y_1) + \cdots + a\beta(y_n) \\ &= a \cdot 1_{\mathtt{triv}} + \cdots + a \cdot 1_{\mathtt{triv}} = n(a \cdot 1_{\mathtt{triv}}). \end{aligned}$$

If $p \mid n$ then $1_{\mathtt{triv}} = n(a \cdot 1_{\mathtt{triv}}) = 0$ and this is a contradiction. Therefore we must have

$p \nmid n$ when the sequence splits. If $p \nmid n$ then $1_{\texttt{triv}} = n(a \cdot 1_{\texttt{triv}}) \implies a = n^{-1}$ and $\delta(1_{\texttt{triv}}) = \frac{1}{n}(y_1 + \cdots + y_n) = \frac{1}{n}Y$. $\square$

When $p \nmid n$, V splits as a direct sum $V \cong \mathfrak{h} \oplus \Bbbk Y$ with irreducible factors. Therefore

$$0 \subset \mathfrak{h} \subset V$$

is a composition series for V where $V/\mathfrak{h} \cong \texttt{triv}$ and $\mathfrak{h} \cong \texttt{stand}$.
When $p \mid n$, for all $(n,p) \neq (2,2)$

$$0 \subset \Bbbk Y \subset \mathfrak{h} \subset V$$

is a composition series for V with irreducible factors $V/\mathfrak{h} \cong \texttt{triv}$, $\mathfrak{h}/\Bbbk Y$ and $\Bbbk Y \cong \texttt{triv}$. When $(n,p) = (2,2)$ we have $\mathfrak{h} = \Bbbk Y$ so $\mathfrak{h}/\Bbbk Y = 0$ and

$$0 \subset \mathfrak{h} \subset V$$

is a composition series for V with irreducible factors $V/\mathfrak{h} \cong \texttt{triv}$ and $\mathfrak{h} \cong \texttt{triv}$.

It is standard practice that the reflection representation chosen should be an irreducible representation of the reflection group. However, for the sake of consistency we will fix the reflection representation of S_n to be $\mathfrak{h}$ in all characteristics. Different authors make different choices for which reflection representation to use, with some authors preferring to use the representation we denote V and others using what we denote by $\mathfrak{h}$. Despite these differences, in characteristic $p \nmid n$ it is possible to translate results between conventions and the representation theory does not depend on this choice, as explained in Section 4.2.

3.2 The dual picture

Let V^* be the dual representation with dual basis $\{x_1, \ldots, x_n\}$, so that $\langle x_i, y_j \rangle = \delta_{ij}$ where $\langle \cdot, \cdot \rangle$ denotes natural pairing $V^* \otimes V \to \Bbbk$.

Lemma 3.2.1. *The representations V and V^* are isomorphic representations of S_n.*

Proof. The map $\phi : V \to V^*$ defined by $\phi(y_k) = x_k$ for all $k \in \{1, \ldots, n\}$ is an isomorphism of vector spaces because V and V^* have equal finite dimension. The map ϕ is an isomorphism of representations if it commutes with the action of S_n and it is sufficient to show this with generators and a basis.
Let $i, j \in \{1, \ldots, n\}$, $i < j$ so that $(ij) \in S_n$ is a generator, and let $k \in \{1, \ldots, n\}$ so y_k is an arbitrary basis vector in $\mathfrak{h}$. For $g \in S_n$ the action on the dual space V^* is defined by $\langle g.x, y \rangle = \langle x, g^{-1}.y \rangle$ for all $y \in \mathfrak{h}$, $x \in \mathfrak{h}^*$. We will show that $\phi\big((ij).y_k\big) = (ij).\phi(y_k)$.

Using properties of dual bases we can write

$$(ij).x_k = \sum_{l=1}^{n} \langle (ij).x_k, y_l \rangle x_l.$$

By the definition of the dual representation we have

$$(ij).x_k = \sum_{l=1}^{n} \langle x_k, (ij).y_l \rangle x_l$$

noting that transpositions are self-inverse. The coefficient $\langle x_k, (ij).y_l \rangle$ of x_l is 1 if and only if $(ij).y_l = y_k$, otherwise the coefficient is 0.
Suppose that $(ij).y_k = y_k$. It then follows from the above expression that $(ij).x_k = x_k$. Hence

$$\phi\big((ij).y_k\big) = \phi(y_k) = x_k = (ij).x_k = (ij).\phi(y_k).$$

Now suppose that $(ij).y_k \neq y_k$. Therefore either $k = i$ or $k = j$. Note that

$$(ij).x_i = \sum_{l=1}^{n} \langle x_i, (ij).y_l \rangle x_l = x_j$$

and it follows that $(ij).x_j = x_i$. Now if $k = i$ then

$$\phi\big((ij).y_k\big) = \phi(y_j) = x_j = (ij).x_i = (ij).\phi(y_k).$$

Similarly if $k = j$ then

$$\phi\big((ij).y_k\big) = \phi(y_i) = x_i = (ij).x_j = (ij).\phi(y_k).$$

□

Since V and V^* are isomorphic representations, Lemma 3.1.1 implies that V^* also has exactly two subrepresentations (one trivial and one standard), and Lemma 3.1.2 implies that V^* decomposes as a direct sum of its two subrepresentations if and only if $p \nmid n$. By exactness of the dual functor we have a short exact dual sequence

$$0 \longrightarrow \mathtt{triv}^* \xrightarrow{\beta^*} V^* \xrightarrow{\alpha^*} \mathfrak{h}^* \longrightarrow 0$$

where $\mathtt{triv}^*$ has a dual basis $\{1_{\mathtt{triv}^*}\}$ with the property $\langle 1_{\mathtt{triv}^*}, 1_{\mathtt{triv}} \rangle = 1$. By short exactness, β^* is an inclusion map and we will determine the image of β^* by applying $\beta^*(1_{\mathtt{triv}^*})$ to the basis of V. Since

$$\beta^*(1_{\mathtt{triv}^*})(y_i) = \langle 1_{\mathtt{triv}^*}, \beta(y_i) \rangle = \langle 1_{\mathtt{triv}^*}, 1_{\mathtt{triv}} \rangle = 1$$

for every $i \in \{1, \ldots, n\}$ this tells us that $\beta^*(1_{\texttt{triv}^*}) = x_1 + x_2 + \cdots + x_n$. As representations, $\texttt{triv}^* \cong \texttt{triv}$ and we know that V^* has a trivial subrepresentation. In fact β^* maps $\texttt{triv}^*$ to its isomorphic copy inside of V^* which is given by the trivial subrepresentation $\Bbbk X$ where $X = x_1 + \cdots + x_n \in V^*$. We can now write the short exact dual sequence as

$$0 \longrightarrow \Bbbk X \xrightarrow{\beta^*} V^* \xrightarrow{\alpha^*} \mathfrak{h}^* \longrightarrow 0.$$

The representation $\mathfrak{h}^* = V^*/\Bbbk X$ is a quotient representation and it is dual to $\mathfrak{h}$ with dual basis $\{\overline{x_1}, \ldots, \overline{x_{n-1}}\}$, where $\overline{x_i}$ is the image of x_i under the quotient map $\alpha^* : V^* \twoheadrightarrow \mathfrak{h}^*$ for every $i \in \{1, \ldots, n\}$. Furthermore, the above sequence splits under the same conditions as V, that is, when $p \nmid n$ we have $V^* \cong \Bbbk X \oplus \mathfrak{h}^*$.

In general, if we define $\mathfrak{h}$ as a subrepresentation of V then consequently $\mathfrak{h}^*$ is a quotient of V^* because the dual functor inverts the direction of arrows in a short exact sequence. We can swap the roles of V and V^* with little consequence, provided that $\mathfrak{h}$ and $\mathfrak{h}^*$ are still defined with one as a subrepresentation and the other a quotient. However in characteristic $p \nmid n$, when V and V^* split as direct sums, we may take the perspective that both $\mathfrak{h}$ and $\mathfrak{h}^*$ are subrepresentations.

Proposition 3.2.2. *Suppose $p \nmid n$. The map $\pi : \mathfrak{h}^* \to V^*$ defined by*

$$\pi(\overline{x_i}) = x_i - \frac{X}{n}$$

for all $i \in \{1, \ldots, n\}$ is the section of the map α^* in the in the split exact sequence

$$0 \longrightarrow \Bbbk X \xrightarrow{\beta^*} V^* \xrightarrow{\alpha^*} \mathfrak{h}^* \longrightarrow 0$$

which realises $\mathfrak{h}^*$ as a subrepresentation of V^*.

Proof. When $p \nmid n$ the above sequence splits, therefore it is right-split and there is a section of α^* we call $\pi : \mathfrak{h}^* \to V^*$ with the property that $\alpha^* \circ \pi$ is an identity map. The map α^* is the quotient of V^* by $\Bbbk X$ therefore for any constant $\lambda \in \Bbbk$ we have $\alpha^*(x_i + \lambda X) = \overline{x_i}$. We require $\alpha^*(\pi(\overline{x_i})) = \overline{x_i}$ for every $i \in \{1, \ldots, n\}$ so let $\pi(\overline{x_i}) = x_i + a_i X$ for some constants $a_i \in \Bbbk$. Since π is a map of representations, it commutes with the action of S_n. Take $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and observe that

$$\begin{aligned} \pi(\overline{x_j}) &= \pi\big((ij).\overline{x_i}\big) \\ &= (ij).\big(\pi(\overline{x_i})\big) \\ &= (ij).\big(x_i + a_i X)\big) \\ &= x_j + a_i X \end{aligned}$$

which implies the constants $a_1 = a_2 = \cdots = a_n$ are all equal. Let $a \in \Bbbk$ be this constant so that for all $i \in \{1, \dots, n\}$ we have $\pi(\overline{x_i}) = x_i + aX$. Now to determine this constant, see that

$$\begin{aligned} 0 = \pi(0) &= \pi(\overline{x_1} + \overline{x_2} + \cdots + \overline{x_n}) \\ &= (x_1 + aX) + (x_2 + aX) + \cdots + (x_n + aX) \\ &= X + (na)X \end{aligned}$$

which shows that $na = -1$ and $a = -\frac{1}{n}$. Therefore the map $\pi : \mathfrak{h}^* \to V^*$ defined for all $i \in \{1, \dots, n\}$ by $\pi(\overline{x_i}) = x_i - \frac{X}{n}$ is the section of the quotient map α^* and realises $\mathfrak{h}^*$ as a subrepresentation of V^* □

In characteristic $p \nmid n$, the map π allows us to perform calculations in $\mathfrak{h}^*$ more easily because we can lift the calculations to V^* and then take a quotient to $\mathfrak{h}^*$ afterwards. In this way, we need not think of $\mathfrak{h}^*$ as a quotient and thus do not have to be mindful of X-cosets in our calculations. However, in characteristic $p \mid n$ we do not have this luxury.

Since V and V^* are isomorphic representations they have the same composition series. When $p \nmid n$, V^* splits as a direct sum $V^* \cong \Bbbk X \oplus \mathfrak{h}^*$ with irreducible factors. Therefore

$$0 \subset \mathfrak{h}^* \subset V^*$$

is a composition series for V^* where $V^*/\mathfrak{h}^* \cong \texttt{triv}$ and $\mathfrak{h}^* \cong \texttt{stand}$. When $p \mid n$, for all $(n, p) \neq (2, 2)$

$$0 \subset \Bbbk X \subset W \subset V^*$$

is a composition series for V^* with irreducible factors $V^*/W \cong \texttt{triv}$, $W/\Bbbk X$ and $\Bbbk X \cong \texttt{triv}$ where $W = \left\{ \sum b_i x_i \mid \sum b_i = 0 \right\}$ is an irreducible representation isomorphic to $\mathfrak{h}$. When $(n, p) = (2, 2)$

$$0 \subset \mathfrak{h}^* \subset V^*$$

is a composition series for V^* with irreducible factors $V^*/\mathfrak{h}^* \cong \texttt{triv}$ and $\mathfrak{h}^* \cong \texttt{triv}$.

3.3 Symmetric polynomials

In order to construct a baby Verma module it is important for us to determine the invariants of our chosen reflection group S_n. If the invariants form a *polynomial algebra* then the Hilbert series of the algebra of invariants is easy to describe based on the degrees of its generators. Consequently, the Hilbert series of the baby Verma module is also easy to describe. In this section we will describe the algebras of S_n invariants in both $S(V^*)$ and $S(\mathfrak{h}^*)$ for all n and p, and summarise the results in Proposition 3.3.12.

In certain characteristics, a theorem due to Claude Chevalley [Ch55], and Shephard–Todd [ShTo54] gives the necessary condition for the algebra of invariants to be polynomial.

Theorem 3.3.1 (Chevalley–Shephard–Todd)**.** *Let W be a vector space over a field $\Bbbk$ and let $G \leq GL(W)$ be a subgroup with char $\Bbbk \nmid |G|$. The algebra of invariants $S(W^*)^G$ is a polynomial algebra if and only if G is a reflection group with reflection representation W.*

The Chevalley–Shephard–Todd theorem tell us that the algebras of invariants $S(V^*)^{S_n}$ and $S(\mathfrak{h}^*)^{S_n}$ in characteristic $p > n$ are polynomial because $|S_n| = n!$ has divisors $1, 2, \ldots, n$ and S_n is a reflection group on V and $\mathfrak{h}$.

Definition 3.3.2. The *elementary symmetric polynomial of degree d* is

$$\widetilde{\sigma_d} = \sum_{1 \leq i_1 < i_2 < \cdots < i_d \leq n} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

The elementary symmetric polynomial of degree d is the sum of all monomials with d distinct variables chosen from $x_1, \ldots, x_n$. In particular, $\widetilde{\sigma_1} = \sum_{i=1}^n x_i = X$ and $\widetilde{\sigma_n} = \prod_{i=1}^n x_i$.

Example 3.3.3. The elementary symmetric polynomials for $n = 2$ are

$$\widetilde{\sigma_1} = x_1 + x_2,$$
$$\widetilde{\sigma_2} = x_1 x_2,$$

and $\widetilde{\sigma_d} = 0$ for all $d \geq 3$.

Example 3.3.4. The elementary symmetric polynomials for $n = 3$ are

$$\widetilde{\sigma_1} = x_1 + x_2 + x_3,$$
$$\widetilde{\sigma_2} = x_1 x_2 + x_1 x_3 + x_2 x_3,$$
$$\widetilde{\sigma_3} = x_1 x_2 x_3,$$

and $\widetilde{\sigma_d} = 0$ for all $d \geq 4$.

Theorem 3.3.5 (The Fundamental Theorem of Symmetric Polynomials)**.** *For all n, and for all field characteristics, the algebra of invariants $(SV^*)^{S_n}$ is a polynomial algebra with homogeneous generators $\widetilde{\sigma_1}, \widetilde{\sigma_2}, \ldots, \widetilde{\sigma_n}$.*

Proof. This is a well-known result, with the classical proof due to Gauss. For an alternative proof see [BlCo17]. □

The Fundamental Theorem of Symmetric Polynomials concludes that any symmetric polynomial can always be expressed as a polynomial in the elementary symmetric polynomials $\widetilde{\sigma_1}, \ldots \widetilde{\sigma_n}$ in a unique way and that this holds over fields of arbitrary characteristic. This can be written as $(SV^*)^{S_n} \cong \Bbbk[\widetilde{\sigma_1}, \ldots, \widetilde{\sigma_n}]$.

Remark 3.3.6. In good characteristic, there are multiple choices of the generators of $(SV^*)^{S_n}$. For example instead of the elementary symmetric polynomial $\widetilde{\sigma_3}$ we could choose the so-called *power sum*

$$x_1^3 + x_2^3 + \cdots + x_n^3$$

as a generator in degree 3. However, in characteristic $p = 3$ we have

$$x_1^3 + x_2^3 + \cdots + x_n^3 = (x_1 + x_2 + \cdots + x_n)^3 = \widetilde{\sigma_1}^3$$

which shows that some choices of generators may fail to be algebraically independent in all characteristics.

The Fundamental Theorem of Symmetric Polynomials (FTSP) tells us that over fields of any characterstic, the invariants of $S(V^*)$ are polynomial and the elementary symmetric polynomials of degrees $1, 2, \ldots, n$ are a good choice of generators. We will now consider the representation $\mathfrak{h}^*$ to determine the invariants of $S(\mathfrak{h}^*)$ and whether or not they form a polynomial algebra. We begin with the following result of Haruhisa Nakajima.

Proposition 3.3.7 ([Na79], Proposition 4.1)**.** *Let U and W be faithful representations of a reflection group G and suppose there exists an epimorphism $U \to W$. If $S(U)^G$ is polynomial then $S(W)^G$ is polynomial.*

By FTSP, the algebra of invariants $S(V^*)^{S_n}$ is polynomial. The representation $\mathfrak{h}^*$ is a quotient of V^* therefore there is an epimorphism $V^* \twoheadrightarrow \mathfrak{h}^*$. Hence to determine when $S(\mathfrak{h}^*)^{S_n}$ is polynomial we only need to determine when V^* and $\mathfrak{h}^*$ are faithful representations of S_n.

Lemma 3.3.8. *The symmetric group S_n acts faithfully on V^*.*

Proof. Let σ be any element of S_n other than the identity. There exists $i \in \{1, \ldots, n\}$ such that $\sigma(i) \neq i$, hence $\sigma.x_i \neq x_i$. □

Lemma 3.3.9. *The symmetric group S_n acts faithfully on $\mathfrak{h}^*$ unless $(n, p) = (2, 2)$.*

Proof. Recall $\mathfrak{h}^*$ is the quotient of V^* by $\Bbbk X = \text{span}_{\Bbbk}\{x_1 + \cdots + x_n\}$, with a basis $\{\overline{x_1}, ..., \overline{x_{n-1}}\}$. Let σ be any element of S_n other than the identity.

Case 1 Suppose that there exists $i \in \{1, \ldots, n\}$ such that $\sigma(i) \neq i$ with both $i \neq n$ and $\sigma(i) \neq n$. Now $\sigma.\overline{x_i} = \overline{x_{\sigma(i)}} \neq \overline{x_i}$ which shows that S_n acts faithfully on $\mathfrak{h}^*$.

Case 2 If such an i does not exist, then there exists an $i \in \{1, \ldots, n-1\}$ such that $\sigma = (in)$. In $\mathfrak{h}^*$ we have $\overline{x_1} + \overline{x_2} + \cdots + \overline{x_n} = 0$ therefore $\overline{x_n} = -\overline{x_1} - \overline{x_2} - \cdots - \overline{x_{n-1}}$. Now

$$\sigma.\overline{x_i} = \overline{x_n} = -\overline{x_1} - \overline{x_2} - \cdots - \overline{x_{n-1}}$$

which is different to $\overline{x_i}$ unless $(n, p) = (2, 2)$. □

Although S_2 does not act faithfully on $\mathfrak{h}^*$ in characteristic 2 we can still determine its invariants in this case.

Lemma 3.3.10. *In characteristic 2, the algebra of invariants $S(\mathfrak{h}^*)^{S_2}$ is a polynomial algebra generated by $\overline{x_1}$.*

Proof. When $n = 2$ and $p = 2$, we define $\mathfrak{h}^*$ as the quotient of $V^* = \text{span}_{\Bbbk}\{x_1, x_2\}$ by the submodule $\Bbbk X = \text{span}_{\Bbbk}\{x_1 + x_2\}$, with basis $\{\overline{x_1}\}$ and $\overline{x_1} + \overline{x_2} = 0$. Now

$$(12).\overline{x_1} = \overline{x_2} = -\overline{x_1} = \overline{x_1}$$

therefore $\overline{x_1}$ is an S_2 invariant of $S(\mathfrak{h}^*)$. Since $S(\mathfrak{h}^*)$ is generated by $\overline{x_1}$ we have shown that $S(\mathfrak{h}^*)^{S_2} = S(\mathfrak{h}^*)$, hence the algebra of invariants is polynomial. □

From these lemmas, we can conclude that the algebra of invariants $S(\mathfrak{h}^*)^{S_n}$ is always polynomial. The following result of Gregor Kemper shall be used to determine the generators of $S(\mathfrak{h}^*)^{S_n}$.

Proposition 3.3.11 ([Ke96], Proposition 16)**.** *Let G be a finite group and W be a faithful representation of G of dimension m over $\Bbbk$. Suppose $f_1, \dots, f_m \in S(W)^G$ are homogeneous invariants of degrees $d_1, \dots, d_m$. These invariants are generators of $S(W)^G$ if and only if $f_1, \dots, f_m$ are algebraically independent over $\Bbbk$ and $\prod_{i=1}^m d_i = |G|$.*

Denote by $\overline{\sigma_i}$ the image of the elementary symmetric polynomial $\widetilde{\sigma_i}$ under the induced quotient map $S(V^*) \twoheadrightarrow S(\mathfrak{h}^*)$. Since $\overline{\sigma_2}, \dots, \overline{\sigma_n}$ are algebraically independent homogeneous invariants of degrees $2, \dots, n$ it follows from Kemper's Proposition that these quotients of elementary symmetric polynomials are the generators of $S(\mathfrak{h}^*)^{S_n}$ for all $(n,p) \neq (2,2)$ because $|S_n| = n!$ which is the product of the degrees of $\overline{\sigma_2}, \overline{\sigma_3}, \dots, \overline{\sigma_n}$.

Proposition 3.3.12. *The algebras of invariants $S(V^*)^{S_n}$ and $S(\mathfrak{h}^*)^{S_n}$ are always polynomial algebras with generators given by the following table.*

Algebra of invariants	$\boldsymbol{n, p}$	**Polynomial invariants**	**Generators**	**Hilbert series**
$S(V^*)^{S_n}$	all n, all p	yes	$\widetilde{\sigma_1}, \widetilde{\sigma_2}, \ldots, \widetilde{\sigma_n}$	$\dfrac{1}{(1-z)(1-z^2)\cdots(1-z^n)}$
$S(\mathfrak{h}^*)^{S_n}$	$(n,p) = (2,2)$	yes	$\overline{x_1}$	$\dfrac{1}{1-z}$
	$(n,p) \neq (2,2)$	yes	$\overline{\sigma_2}, \ldots, \overline{\sigma_n}$	$\dfrac{1}{(1-z^2)\cdots(1-z^n)}$

Table 3.3.A: Algebras of invariants, and their generators and Hilbert series.

In characteristic $p \nmid n$, we have a map $\pi : \mathfrak{h}^* \to V^*$ which realises $\mathfrak{h}^*$ as a subrepresentation of V^* as explained in Proposition 3.2.2. In this case, the map extends naturally to a map $S(\mathfrak{h}^*) \to S(V^*)$ and we define by $\sigma_i = \pi(\overline{\sigma_i})$ the symmetric polynomials $\sigma_2, \sigma_3, \ldots, \sigma_n$. We use $\{\sigma_2, \ldots, \sigma_n\}$ as generators of $S(\mathfrak{h}^*)^{S_n}$ when $p \nmid n$.

It should be noted that the invariants of $S(\mathfrak{h}^*)$ are special, because $\mathfrak{h}^*$ is defined as a quotient of V^* and we can therefore use the work of Nakajima to conclude that the invariants are polynomial. If we were instead to consider the algebra of invariants $S(\mathfrak{h})^{S_n}$ then since $\mathfrak{h}$ is a subrepresentation of V and not a quotient, Nakajima's result does not apply and it turns out that these invariants fail to be polynomial in all cases.

Example 3.3.13 ([KeMa97], Corollary 5.2)**.** Suppose $p \mid n$, and $n \geq 5$. The algebra of invariants $S(\mathfrak{h})^{S_n}$ is not polynomial.

Chapter 4

The Rational Cherednik Algebra of The Symmetric Group

In this chapter we will restate the definitions and theorems from Chapter 2 in the case of the reflection group $G = S_n$. We will also explain the discrepancies which may arise depending on the choice of reflection representation.

4.1 The type A_{n-1} rational Cherednik algebra

The symmetric group S_n is a Coxeter group of type A_{n-1}. For this reason the rational Cherednik algebra corresponding to the symmetric group is also called the type A_{n-1} rational Cherednik algebra. We fix our reflection group G to be the symmetric group S_n. Let $V = \operatorname{span}_{\Bbbk}\{y_1, \dots, y_n\}$ be the permutation representation of S_n and denote its dual representation by $V^* = \operatorname{span}_{\Bbbk}\{x_1, \dots, x_n\}$ with a dual basis so that $\langle x_i, y_j\rangle = \delta_{ij}$ where $\langle \cdot, \cdot \rangle$ denotes natural pairing $V^* \otimes V \to \Bbbk$ and

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

For the reflection representation of S_n, we fix the subrepresentation

$$\mathfrak{h} = \left\{ \sum_{i=1}^{n} a_i y_i \in V \;\middle|\; \sum_{i=1}^{n} a_i = 0 \right\} \subset V$$

of V with basis $\{\, y_1 - y_n,\, y_2 - y_n,\, \dots,\, y_{n-1} - y_n \,\}$ and spanning set $\{\, y_i - y_j \mid i \neq j \,\}$. The dual of $\mathfrak{h}$ is the quotient representation $\mathfrak{h}^* = V^*/\Bbbk X$ where $X = x_1 + \cdots + x_n$. The pairing $\langle \cdot, \cdot \rangle$ descends to $\mathfrak{h}^* \otimes \mathfrak{h}$ such that $\mathfrak{h}^*$ has a dual basis $\{\overline{x_1}, \dots, \overline{x_{n-1}}\}$ where $\overline{x_i}$ the image of x_i under the quotient map $\alpha^* : V^* \twoheadrightarrow \mathfrak{h}^*$ for every $i \in \{1, \dots, n\}$. The reflections in S_n are the transpositions (ij) for $1 \leq i < j \leq n$. By Proposition 1.1.7, for each $s = (ij)$ there is a unique element $\alpha_s \otimes \alpha_s^\vee \in \mathfrak{h}^* \otimes \mathfrak{h}$ where α_s and $\alpha_s^\vee$ span the images of $(1 - (ij))$ in $\mathfrak{h}^*$ and $\mathfrak{h}$

respectively. For the symmetric group, we pick $\alpha_{(ij)} = \overline{x_i} - \overline{x_j}$ and calculate $\alpha^\vee_{(ij)} = y_i - y_j$. All the reflections in S_n belong to the same conjugacy class, so the parameter c is some element of the field $\Bbbk$. We also choose some $t \in \Bbbk$. We construct the Cherednik algebra $H_{t,c}(S_n, \mathfrak{h})$ as the quotient of $\Bbbk[S_n] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = \langle x, y\rangle t - c \sum_{(ij)\in S} \langle (1 - (ij)).x, y\rangle (ij)$$

for all $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$. For each irreducible representation $\tau \in \widehat{S_n}$ we can construct the Verma module $M_{t,c}(\tau)$ which is $S(\mathfrak{h}^*) \otimes \tau$ as a vector space. Given $f \otimes v \in S(\mathfrak{h}^*) \otimes \tau$ the action of the algebra is given by

$$\begin{aligned}
x.(f \otimes v) &= (xf) \otimes v,\\
g.(f \otimes v) &= (g.f) \otimes (g.v),\\
y.(f \otimes v) &= t\partial_y(f) \otimes v - c \sum_{(ij)\in S} \langle \overline{x_i} - \overline{x_j}, y\rangle \frac{(1 - (ij)).f}{\overline{x_i} - \overline{x_j}} \otimes (ij).v
\end{aligned}$$

for any $x \in \mathfrak{h}^*$, $g \in S_n$, and $y \in \mathfrak{h}$.
By Proposition 3.3.12, for all $(n, p) \neq (2, 2)$ the generators of the invariant subalgebra $S(\mathfrak{h}^*)^{S_n}$ are the quotients of elementary symmetric polynomials, $\overline{\sigma_2}, \overline{\sigma_3}, \dots, \overline{\sigma_n}$, which have degrees 2, $3, \dots, n$. Using the formulas given in Examples 2.6.9 and 2.6.10, we obtain the following.

Example 4.1.1. When $(n, p) \neq (2, 2)$, the Hilbert polynomial of the baby Verma module $N_{0,c}(S_n, \mathfrak{h}, \tau)$ is

$$\mathrm{Hilb}_{N_{0,c}(S_n,\mathfrak{h},\tau)}(z) = (1 - z^2)(1 - z^3)\cdots(1 - z^n)\frac{\dim \tau}{(1 - z)^{n-1}} .$$

Example 4.1.2. When $(n, p) \neq (2, 2)$, the Hilbert polynomial of the baby Verma module $N_{1,c}(S_n, \mathfrak{h}, \tau)$ is

$$\mathrm{Hilb}_{N_{1,c}(S_n,\mathfrak{h},\tau)}(z) = (1 - z^{2p})(1 - z^{3p})\cdots(1 - z^{np})\frac{\dim \tau}{(1 - z)^{n-1}} .$$

The only non-trivial linear character of S_n is `sign` which acts on reflections by -1. Since $\mathtt{sign} = \mathtt{sign} \otimes \mathtt{triv}$, the character of $L_{t,c}(\mathtt{sign})$ can be calculated from the character of $L_{t,-c}(\mathtt{triv})$ using the formula in Corollary 2.8.3,

$$\chi_{L_{t,c}(\mathtt{sign})} = \chi_{L_{t,-c}(\mathtt{triv})} \cdot [\mathtt{sign}].$$

Proposition 4.1.3. *Consider the rational Cherednik algebra* $H_{t,c}(S_n, \mathfrak{h})$.

1. *If* $t = 0$, *all* $c \neq 0$ *are generic.*
2. *If* $t = 1$, *all* $c \notin \mathbb{F}_p$ *are generic.*

Proof. By Proposition 2.1.2, $H_{0,c}(S_n, \mathfrak{h}) \cong H_{0,\lambda c}(S_n, \mathfrak{h})$ for any nonzero $\lambda \in \Bbbk$, hence all nonzero c are generic at $t = 0$. At $t = 1$ this is Proposition 2.8 in [Li14]. □

Let us rephrase Theorem 3.2 from [Li14] here in our conventions (as [Li14] works with the reflection representation V and we work with $\mathfrak{h}$) and state a slight strengthening of it with the same proof.

Proposition 4.1.4 ([Li14], Theorem 3.2)**.** *Let* $\Bbbk$ *be an algebraically closed field of characteristic* p, *and consider the rational Cherednik algebra* $H_{t,c}(S_n, \mathfrak{h})$. *The character of the irreducible representation* $L_{t,c}(\texttt{triv})$ *is*

$$\chi_{L_{t,c}(\texttt{triv})}(z) = [\texttt{triv}]$$

if and only if $t = cn$. *In particular, this holds when* $t = c = 0$, *and when* p *does not divide* n *and the values of the parameters are* $t = 1, c = 1/n$.

Proof. It suffices to note that

$$\begin{aligned} D_{y_i - y_j}(\overline{x_k}) &= t\partial_{y_i - y_j}(\overline{x_k}) - c\left(\sum_{a<b} \langle y_i - y_j, \overline{x_a} - \overline{x_b}\rangle \frac{\overline{x_k} - (ab).\overline{x_k}}{\overline{x_a} - \overline{x_b}}\right) \\ &= t(\delta_{ik} - \delta_{jk}) - c\left(\sum_{a} \langle y_i - y_j, \overline{x_k} - \overline{x_a}\rangle\right) \\ &= (t - cn)(\delta_{ik} - \delta_{jk}). \end{aligned}$$

This shows that all $\overline{x_k} \in M^1_{t,c}(\texttt{triv})$ are singular if and only if $t = cn$.

□

4.2 Moving between V and $\mathfrak{h}$

In characteristic $p \nmid n$, when $\mathfrak{h}$ is a direct summand of V, the character of the irreducible object $L_{t,c}(S_n, \mathfrak{h}, \tau)$ in category $\mathcal{O}$ for $H_{t,c}(S_n, \mathfrak{h})$ can be derived from the irreducible object $L_{t,c}(S_n, V, \tau)$ in category $\mathcal{O}$ for $H_{t,c}(S_n, V)$. Therefore whether one chooses V or $\mathfrak{h}$ as the reflection representation of S_n has no real impact, but here we describe how to translate results between conventions.

Lemma 4.2.1. *Let* $i \in \{1, \ldots, n\}$. *The following sums are equal.*

$$\sum_{s\in S} \langle (1-s).x_i, y_i\rangle s = \sum_{\substack{k=1,\\ k\neq i}}^{n} (ik)$$

Proof. For each $s \in S$, observe that the pairing $\langle x_i - s.x_i, y_i \rangle$ is zero when $s.x_i = x_i$ and 1 otherwise. The reflections $s \in S$ such that $s.x_i \neq x_i$ are all reflections of the form $s = (ik)$ where $k \in \{1, \dots, n\}$ and $k \neq i$. □

Lemma 4.2.2. *Let $i, j \in \{1, \dots, n\}$ and suppose $i \neq j$. The following equality holds.*

$$\sum_{s \in S} \langle (1-s).x_i, y_j \rangle s = -(ij)$$

Proof. Observe that the pairing $\langle x_i - s.x_i, y_j \rangle$ is only nonzero for the reflection $s = (ij)$, and $\langle x_i - x_j, y_j \rangle = -1$. □

Lemma 4.2.3. *For all $i, j \in \{1, \dots, n\}$ and all $d \geq 1$ we have the following commutation relations in $H_{t,c}(S_n, V)$.*

1. $[Y, x_i - x_j] = 0$.
2. $[y_i, X] = t$.
3. $[y_i, X^d] = tdX^{d-1}$.

where $Y = y_1 + \cdots + y_n$ and $X = x_1 + \cdots + x_n$.

Proof. The proofs are by direct calculation which we now show, while making use of the identities shown in the previous two lemmas.
Proof of 1.

$$\begin{aligned}
[Y, x_i - x_j] &= \sum_{k=1}^{n} [y_k, x_i] - \sum_{k=1}^{n} [y_k, x_j] \\
&= \sum_{k=1}^{n} \left(\langle x_i, y_k \rangle t - c \sum_{s \in S} \langle (1-s).x_i, y_k \rangle s \right) \\
&\quad - \sum_{k=1}^{n} \left(\langle x_j, y_k \rangle t - c \sum_{s \in S} \langle (1-s).x_j, y_k \rangle s \right) \\
&= \sum_{\substack{k=1, \\ k \neq i}}^{n} \left(\langle x_i, y_k \rangle t - c \sum_{s \in S} \langle (1-s).x_i, y_k \rangle s \right) + \langle x_i, y_i \rangle t - c \sum_{s \in S} \langle (1-s).x_i, y_i \rangle s \\
&\quad - \sum_{\substack{k=1, \\ k \neq j}}^{n} \left(\langle x_j, y_k \rangle t - c \sum_{s \in S} \langle (1-s).x_j, y_k \rangle s \right) - \langle x_j, y_j \rangle t + c \sum_{s \in S} \langle (1-s).x_j, y_j \rangle s \\
&= \sum_{\substack{k=1, \\ k \neq i}}^{n} \Big(0 + c(ik) \Big) + t - c \sum_{\substack{k=1, \\ k \neq i}}^{n} (ik) - \sum_{\substack{k=1, \\ k \neq j}}^{n} \Big(0 + c(jk) \Big) - t + c \sum_{\substack{k=1, \\ k \neq j}}^{n} (jk) = 0.
\end{aligned}$$

Proof of 2.

$$\begin{aligned}
[y_i, X] &= \sum_{k=1}^{n}[y_i, x_k] \\
&= \sum_{k=1}^{n}\left(\langle x_k, y_i\rangle t - c\sum_{s\in S}\langle(1-s).x_k, y_i\rangle s\right) \\
&= \sum_{\substack{k=1,\\ k\neq i}}^{n}\left(\langle x_k, y_i\rangle t - c\sum_{s\in S}\langle(1-s).x_k, y_i\rangle s\right) \\
&\quad + \langle x_i, y_i\rangle t - c\sum_{s\in S}\langle(1-s).x_i, y_i\rangle s \\
&= \sum_{\substack{k=1,\\ k\neq i}}^{n}\Big(0 + c(ik)\Big) + t - c\sum_{\substack{k=1,\\ k\neq i}}^{n}(ik) = t
\end{aligned}$$

The proof of *3* follows from the proof of *2* by induction. □

In characteristic $p \nmid n$, $\mathfrak{h}^*$ has a direct complement in V^* therefore there is a natural inclusion $S(\mathfrak{h}^*) \hookrightarrow S(V^*)$, and we can consider every $f \in S(\mathfrak{h}^*)\otimes\tau$ as an element in $S(V^*)\otimes\tau$.

Lemma 4.2.4. *Suppose $p \nmid n$ and let $f \in M_{t,c}(S_n, \mathfrak{h}, \tau)$ be a homogeneous vector of positive degree. If f is singular in $M_{t,c}(S_n, \mathfrak{h}, \tau)$ (or some quotient thereof), then f is singular in $M_{t,c}(S_n, V, \tau)$ (or some quotient thereof).*

Proof. Suppose f is singular in $M_{t,c}(S_n, \mathfrak{h}, \tau)$ (or some quotient thereof) so that $D_{y_i-y_j}(f) = 0$ for all i, j. By Lemma 4.2.3, Y commutes with everything in $\mathfrak{h}^*$ and therefore commutes with every element of $S(\mathfrak{h}^*)$. Now $f = \sum f_i \otimes v_i$ for some $f_i \in S(\mathfrak{h}^*)$ and $v_i \in \tau$, so

$$D_Y(f) = Y(\sum f_i \otimes v_i) = \sum Y f_i \otimes v_i = \sum f_i Y \otimes v_i = \sum f_i \otimes Y v_i = 0$$

hence f is in the kernel of the Dunkl operator D_Y. Now for any $i \in \{1, \ldots, n\}$ we have

$$0 = \sum_{\substack{j=1,\\ j\neq i}}^{n} D_{y_i-y_j}(f) + D_Y(f) = D_{ny_i}(f) = nD_{y_i}(f)$$

therefore in characteristic $p \nmid n$ we can conclude that $D_{y_i}(f) = 0$ for all i. Hence f is singular as an element of $M_{t,c}(S_n, V, \tau)$ or some quotient thereof. □

In characteristic $p \nmid n$, the permutation representation V splits as a direct sum $V = \mathfrak{h}\oplus\Bbbk Y$. Consequently the dual representation splits as $V^* = \Bbbk X \oplus \mathfrak{h}^*$ and therefore the symmetric algebra of V^* decomposes as $S(V^*) = \Bbbk[X] \otimes S(\mathfrak{h}^*)$.

Lemma 4.2.5. *Suppose $p \nmid n$ and let $f \in S(V^*) \otimes \tau$ be a homogeneous vector of degree d with*

$$f = \sum_{i=0}^{d} X^{d-i} f_i$$

where $f_i \in S^i(\mathfrak{h}^*) \otimes \tau$ has degree i. If f is singular in $M_{t,c}(S_n, V, \tau)$ (or some quotient thereof), then $D_{y_j - y_l}(f_i) = 0$ for all $i \in \{1, \dots, d\}$ and $j, l \in \{1, \dots, n\}$. That is, if f is singular then the f_i are singular vectors in $M_{t,c}(S_n, \mathfrak{h}, \tau)$ or some quotient thereof.

Proof. Suppose $f \in S(V^*) \otimes \tau$ is singular. For all $j \in \{1, \dots, n\}$ we have

$$\begin{aligned} 0 = D_{y_j}(f) = \sum_{i=0}^{d} y_j X^{d-i} f_i &= \sum_{i=0}^{d} [y_j, X^{d-i}] f_i + \sum_{i=0}^{d} X^{d-i} y_j f_i \\ &= \sum_{i=0}^{d-1} t(d-i) X^{d-i-1} f_i + \sum_{i=0}^{d} X^{d-i} D_{y_j}(f_i) \\ &= \sum_{i=1}^{d} t(d-i+1) X^{d-i} f_{i-1} + \sum_{i=1}^{d} X^{d-i} D_{y_j}(f_i) \\ &= \sum_{i=1}^{d} X^{d-i} \big(D_{y_j}(f_i) + t(d-i+1) f_{i-1} \big), \end{aligned}$$

therefore since $D_{y_j}(f_i) \in S(\mathfrak{h}^*) \otimes \tau$, we have $D_{y_j}(f_i) = -t(d-i+1) f_{i-1}$ for all $i \in \{1, \dots, d\}$. Now for all $i \in \{1, \dots, d\}$, $j, l \in \{1, \dots, n\}$ we have

$$\begin{aligned} D_{y_j - y_l}(f_i) &= D_{y_j}(f_i) - D_{y_l}(f_i) \\ &= -t(d-i+1) f_{i-1} + t(d-i+1) f_{i-1} = 0. \end{aligned}$$

therefore f_i is singular in $M_{t,c}(S_n, \mathfrak{h}, \tau)$ (or some quotient thereof). □

Proposition 4.2.6. *Let $\Bbbk$ be an algebraically closed field whose finite characteristic p does not divide n, τ an irreducible representation of S_n, $L_{t,c}(S_n, V, \tau)$ the irreducible representation with lowest weight τ for the rational Cherednik algebra $H_{t,c}(S_n, V)$ over $\Bbbk$ generated by V, V^*, S_n, and $L_{t,c}(S_n, \mathfrak{h}, \tau)$ the irreducible representation with lowest weight τ for the rational Cherednik algebra $H_{t,c}(S_n, \mathfrak{h})$ over $\Bbbk$ generated by $\mathfrak{h}, \mathfrak{h}^*, S_n$. Their characters are related as follows:*

- *if $t = 0$,*

$$\chi_{L_{0,c}(S_n, V, \tau)}(z) = \chi_{L_{0,c}(S_n, \mathfrak{h}, \tau)}(z)$$

- *if $t = 1$,*

$$\chi_{L_{1,c}(S_n, V, \tau)}(z) = \chi_{L_{1,c}(S_n, \mathfrak{h}, \tau)}(z) \cdot \left(\frac{1 - z^p}{1 - z} \right).$$

Proof. The irreducible representation $L_{0,c}(S_n, \mathfrak{h}, \tau)$ (respectively $L_{1,c}(S_n, \mathfrak{h}, \tau)$) can be written as a quotient of the Verma module $M_{0,c}(S_n, \mathfrak{h}, \tau)$ (respectively $M_{1,c}(S_n, \mathfrak{h}, \tau)$) by a submodule $\langle f_1, f_2, \ldots, f_k \rangle$ generated by some homogeneous vectors f_i of degrees $d_i \in \mathbb{N}$. We can assume without loss of generality that $0 < d_1 \leq d_2 \leq \ldots \leq d_k$, and for every i and all $y \in \mathfrak{h}$ we have $D_y(f_i) \in \langle f_1, \ldots, f_{i-1} \rangle$.

We claim that when $t = 0$

$$L_{0,c}(S_n, V, \tau) \cong M_{0,c}(S_n, V, \tau) / \langle f_1, f_2, \ldots, f_k, X \otimes \tau \rangle$$

where $X \otimes \tau$ denotes $X \otimes v$ for every $v \in \tau$. Respectively, when $t = 1$ we claim

$$L_{1,c}(S_n, V, \tau) \cong M_{1,c}(S_n, V, \tau) / \langle f_1, f_2, \ldots, f_k, X^p \otimes \tau \rangle$$

where $X^p \otimes \tau$ denotes $X^p \otimes v$ for every $v \in \tau$.

First, by Lemma 4.2.4, for every i and all $y \in V$ we have $D_y(f_i) \in \langle f_1, \ldots, f_{i-1} \rangle$.

Next, by Lemma 4.2.3 part (2), when $t = 0$ we have $[y, X] = 0$ for all $y \in V$, and by part (3) when $t = 1$ we have $[y, X^p] = 0$ for all $y \in V$. As a consequence, for $t = 0$ the set $X \otimes \tau$ consists of singular vectors in degree 1, and when $t = 1$ the set $X^p \otimes \tau$ consists of singular vectors in degree p.

For $t = 0$ we set $J_0 = \langle f_1, f_2, \ldots, f_k, X \otimes \tau \rangle$ and respectively for $t = 1$ we set $J_1 = \langle f_1, f_2, \ldots, f_k, X^p \otimes \tau \rangle$. In both cases, the above shows that J_t is a proper submodule of $M_{t,c}(S_n, V, \tau)$, and so $L_{t,c}(S_n, V, \tau)$ is a quotient of $M_{t,c}(S_n, V, \tau)/J_t$.

Finally, let us show that the module $M_{t,c}(S_n, V, \tau)/J_t$ is irreducible. If it is not, then there is a homogeneous vector $v \in M^d_{t,c}(S_n, V, \tau)$, which is not in J_t, and is such that for all $y \in V$ we have $D_y(v) \in J_t$. We can assume without loss of generality that d is the smallest such degree. Write $v \in S(V)^* \otimes \tau \cong \Bbbk[X] \otimes S(\mathfrak{h}^*) \otimes \tau$ as $v = \sum_{i=0}^{d} X^{d-i} v_i$ for $v_i \in S^i(\mathfrak{h}^*) \otimes \tau$. The condition $D_{y_j}(v) \in J_t$ for all $j \in \{1, \ldots, n\}$ then becomes

$$\begin{aligned} D_{y_j}(v) = \sum_{i=0}^{d} y_j X^{d-i} v_i &= \sum_{i=0}^{d} [y_j, X^{d-i}] v_i + \sum_{i=0}^{d} X^{d-i} y_j v_i \\ &= \sum_{i=0}^{d-1} t(d-i) X^{d-i-1} v_i + \sum_{i=0}^{d} X^{d-i} D_{y_j}(v_i) \\ &= \sum_{i=1}^{d} t(d-i+1) X^{d-i} v_{i-1} + \sum_{i=1}^{d} X^{d-i} D_{y_j}(v_i) \\ &= \sum_{i=1}^{d} X^{d-i} \big(D_{y_j}(v_i) + t(d-i+1) v_{i-1} \big) \in J_t. \end{aligned}$$

This can be rewritten as

$$X^{d-i} D_{y_j}(v_i) = -t(d-i-1)X^{d-i}v_{i-1} \pmod{J_t}$$

for all $i \in \{0, \dots, m\}$, $j \in \{1, \dots, n\}$. This implies that for any $j, l \in \{1, \dots n\}$ we have

$$\begin{aligned} X^{d-i} D_{y_j - y_l}(v_i) &= X^{d-i} D_{y_j}(v_i) - X^{d-i} D_{y_l}(v_i) \\ &= -t(d-i-1)X^{d-i}v_{i-1} + t(d-i-1)X^{d-i}v_{i-1} = 0 \pmod{J_t} \end{aligned}$$

and consequently

$$X^{d-i} D_y(v_i) \in J_t \quad \text{for all } y \in \mathfrak{h},\ i \in \{0, \dots, d\}.$$

We now consider two cases:

- Suppose $t = 0$. The summand of $v = \sum_{i=0}^{d} X^{d-i} v_i$ when $i = d$ is $v_d \in S(\mathfrak{h}^*) \otimes \tau$. Since $D_y(v_d) \in \langle f_1, f_2, \dots, f_k \rangle$ for all $y \in \mathfrak{h}$ and $M_{t,c}(S_n, \mathfrak{h}, \tau) / \langle f_1, f_2, \dots, f_k \rangle = L_{0,c}(S_n, \mathfrak{h}, \tau)$ is irreducible, it must be the case that $v_d \in \langle f_1, \dots, f_k \rangle$. However,

$$v = v_d + X \sum_{i=0}^{d-1} X^{d-i-1} v_i \in \langle f_1, \dots, f_n, X \otimes v \rangle = J_0,$$

which contradicts the assumption that $v \notin J_0$.

- Suppose $t = 1$. As for all $y \in \mathfrak{h}$, $i \in \{0, \dots, d\}$ we have

$$X^{d-i} D_y(v_i) \in \langle f_1, \dots, f_k, X^p \otimes \tau \rangle$$

we can in particular conclude that for all i such that $d - i < p$ and all $y \in \mathfrak{h}$ we have

$$D_y(v_i) \in \langle f_1, \dots, f_k \rangle .$$

Since $M_{1,c}(S_n, \mathfrak{h}, \tau) / \langle f_1, f_2, \dots, f_k \rangle = L_{1,c}(S_n, \mathfrak{h}, \tau)$ is irreducible, we conclude that

$$v_i \in \langle f_1, \dots, f_k \rangle , \quad \text{for all } d - p < i \le d$$

But then

$$v = X^p \sum_{i=0}^{d-p} X^{d-p-i} v_i + \sum_{i=d-p+1}^{d} X^{d-i} v_i \in \langle f_1, \dots, f_k, X^p \otimes \tau \rangle = J_1,$$

which contradicts the assumption that $v \notin J_1$.

□

Chapter 5

Survey of Related Work

In this chapter we will review recent work with which this thesis has nonzero intersection. As previously mentioned, there is a choice of reflection representation when constructing the rational Cherednik algebra for S_n. Some authors use the permutation representation V and others use the subrepresentation $\mathfrak{h}$. The algebras $H_{t,c}(S_n, V)$ and $H_{t,c}(S_n, \mathfrak{h})$ are related and in characteristic $p \nmid n$ the characters of irreducible modules for one algebra can be deduced from the other by using the following formulas.

$$\chi_{L_{0,c}(S_n,V,\tau)}(z) = \chi_{L_{0,c}(S_n,\mathfrak{h},\tau)}(z)$$
$$\chi_{L_{1,c}(S_n,V,\tau)}(z) = \chi_{L_{1,c}(S_n,\mathfrak{h},\tau)}(z) \cdot \left(\frac{1-z^p}{1-z}\right)$$

For a detailed discussion of moving between V and $\mathfrak{h}$, see Section 4.2 where we prove these formulas.

5.1 *Representations of rational Cherednik algebras of rank 1 in positive characteristic*, Frédéric Latour. Journal of Pure and Applied Algebra (2005) [La05]

In this paper, Frédéric Latour describes irreducible representations of the rational Cherednik algebras associated to cyclic groups in positive characteristic. The cyclic group $\mathbb{Z}/r\mathbb{Z}$ of order r is a reflection group with cyclic generator s and reflection representation $\mathfrak{h} = \mathrm{span}_{\Bbbk}\{y\}$ with action $s.y = \varepsilon y$ for $\varepsilon \in \Bbbk$ a primitive rth root of unity. Latour considers the representation theory of the associated rational Cherednik algebras over a field of positive characteristic p which is relatively prime to r. This overlaps with our work since the cyclic group $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to the symmetric group S_2.

In Latour's notation, we fix $r = 2$ and consider the cyclic group $\mathbb{Z}/2\mathbb{Z}$ which corresponds in our notation with fixing $n = 2$ and the symmetric group S_2. Since $\varepsilon \in \Bbbk$ is a primitive rth root of unity, we shall fix $\varepsilon = -1$ and therefore the reflection representation of S_2 we use is `sign`. In Latour's work, the field $\Bbbk$ of characteristic p relatively prime to r must have characteristic $p \geq 3$ when considering S_2. Latour denotes by $\mathbf{H}_{t,c_1}$ a rational Cherednik algebra with generators x, y, s and relations

$$s^2 = 1, \quad sx = -xs, \quad sy = -ys, \quad [y, x] = t - c_1 s.$$

This corresponds with our definition of $H_{t,c}(S_2, \mathfrak{h})$ with rescaled parameters, where $c_1 = 2c$, $y = y_1 - y_2$ is the basis of $\mathfrak{h}$, $x = \overline{x_1}$ is the basis of $\mathfrak{h}^*$ and s is the non-identity element of S_2. The two irreducible representations $\tau \in \widehat{S_2}$ in characteristics $p \geq 3$ are $\tau = \texttt{triv}$ and $\tau = \texttt{sign}$; this is captured by Latour's parameter $m \in \{0, 1\}$ with $m = 0$ referring to `triv` and $m = 1$ referring to `sign`. Where Latour writes $(-1)^m$ we would write $\chi_\tau(s)$. The Verma module $M_{1,c}(S_2, \mathfrak{h}, \tau)$ has a basis $\{x^i \otimes v \mid i \geq 0\}$ where $\{v\}$ is a basis of τ.

In the propositions that follow, we can identify the basis element $v_i \in W_m$ with $x^i \otimes v \in S^i(\mathfrak{h}^*) \otimes \tau$. There are two irreducible representations of S_2 in characteristic $p \geq 3$ and so the propositions state that there are two irreducible representations of $H_{t,c}(S_2, \mathfrak{h})$ as expected. We specialise the statements of the following propositions to the specific case we study, although the original statements in Latour's paper are more general.

Proposition 5.1.1 ([La05], Proposition 2.8)**.** *In characteristic $p \geq 3$, for each $m \in \{0, 1\}$ there exists a unique irreducible representation $W_{0,m}$ of $H_{0,c}(S_2, \mathfrak{h})$ with $x^2 = y^2 = 0$ and $s.v = (-1)^m v$ whenever $y.v = 0$. The dimension $D \leq 2$ of $W_{0,m}$ is the smallest positive integer satisfying the equation*

$$0 = 2c(-1)^m \sum_{i=0}^{D-1} (-1)^i. \tag{5.1.2}$$

$W_{0,m}$ has a basis $\{v_0, v_1, \ldots, v_{D-1}\}$, such that

$$\begin{aligned}
&x.v_i = v_{i+1} && \text{for } 0 \leq i \leq D-2,\\
&x.v_{D-1} = 0, &&\\
&y.v_i = \mu_i v_{i-1} && \text{for } 1 \leq i \leq D-1,\\
&y.v_0 = 0, &&\\
&s.v_i = (-1)^m(-1)^i v_i && \text{for } 0 \leq i \leq D-1,
\end{aligned}$$

where $\mu_k = -2c(-1)^m \sum_{i=0}^{k-1} (-1)^i$. Every irreducible representation of $H_{0,c}(S_2, \mathfrak{h})$ on which x^2 and y^2 both act as zero is of the form $W_{0,m}$ for some m.

When $c = 0$, (5.1.2) is always satisfied, for any value of D, so we must have $D = 1$ as the smallest positive integer solution. This matches our results in Example 2.4.2 and Proposition 4.1.4.

If $c \neq 0$, then to satisfy (5.1.2) we require the sum $\sum_{i=0}^{D-1} (-1)^i$ to be zero, and the smallest positive integer value of D for which this happens is $D = 2$. This agrees with our result in Proposition 6.1.5.

The irreducible modules described by this proposition are those in category $\mathcal{O}$, as can be seen by the condition $x^2 = y^2 = 0$. An irreducible module in category $\mathcal{O}_{0,c}$ is either a baby Verma module $N_{0,c}(\tau)$, or a quotient of a baby Verma module. The S_2-invariant in $S(\mathfrak{h}^*)$ is x^2 therefore $x^2 = 0$ in any irreducible module in category $\mathcal{O}_{0,c}$ because this invariant is annihilated when taking the quotient to the baby Verma module. The baby Verma module $N_{0,c}(\tau)$ is 2-dimensional and $\mathbb{Z}$-graded, therefore $y^2 = 0$ in every quotient of the baby Verma module $N_{0,c}(\tau)$ as y reduces the degree by 1 and acts as zero in the lowest degree.

Proposition 5.1.3 ([La05], Proposition 2.5)**.** *In characteristic $p \geq 3$, for each $m \in \{0, 1\}$ there exists a unique irreducible representation $W_{1,m}$ of $H_{1,c}(S_2, \mathfrak{h})$ with $x^{2p} = y^{2p} = 0$ and $s.v = (-1)^m v$ whenever $y.v = 0$. The dimension $D \leq 2p$ of $W_{1,m}$ is the smallest positive integer satisfying the equation*

$$D = 2c(-1)^m \sum_{i=0}^{D-1} (-1)^i. \tag{5.1.4}$$

$W_{1,m}$ has a basis $\{v_0, v_1, \ldots, v_{D-1}\}$, such that

$$\begin{aligned}
&x.v_i = v_{i+1} && \text{for } 0 \leq i \leq D-2,\\
&x.v_{D-1} = 0,\\
&y.v_i = \mu_i v_{i-1} && \text{for } 1 \leq i \leq D-1,\\
&y.v_0 = 0,\\
&s.v_i = (-1)^m (-1)^i v_i && \text{for } 0 \leq i \leq D-1,
\end{aligned}$$

where $\mu_k = k - 2c(-1)^m \sum_{i=0}^{k-1} (-1)^i$. Every irreducible representation of $H_{1,c}(S_2, \mathfrak{h})$ on which x^{2p} and y^{2p} both act as zero is of the form $W_{1,m}$ for some m.

The sum $\sum_{i=0}^{D-1} (-1)^i$ is 0 when D is even and 1 when D is odd. So in characteristic p, (5.1.4) is satisfied for every value of c when $D = 2p$ because $2p$ is both even and divisible by p. This observation is found in Remark 2.6 of [La05]. However, for some values of c, a smaller positive integer solution may exist.

If $c = 0$ then (5.1.4) becomes $D = 0$ which is only satisfied when D is divisible by p; since D is the smallest positive integer which satisfies the equation we must have $D = p$. This result follows from Proposition 2.6.13.

Now suppose $c \neq 0$. The dimension D is either even or odd. If D is even, then (5.1.4) becomes $D = 0$ which is only satisfied by $D = 2p$ and we have seen that such solutions do not depend on c. If D is odd then (5.1.4) becomes $D = 2c(-1)^m$ which we shall consider separately for $m = 0$ and $m = 1$.

If $c \notin \mathbb{F}_p$ then no integer $D < 2p$ satisfies (5.1.4), so $D = 2p$. In Proposition 6.2.6, we show that the only solutions strictly smaller than $2p$ are odd, and that such solutions require $c \in \mathbb{F}_p$, so suppose that $c \in \{0, 1, 2, \ldots, p-1\}$. We distinguish two cases, $m = 0$ and $m = 1$, and look for odd D satisfying (5.1.4).
When $m = 0$, (5.1.4) means that we are looking for smallest odd positive integer D which satisfies $D = 2c$ in $\mathbb{F}_p$, in other words the smallest odd positive integer D which satisfies $D \equiv 2c \bmod p$ in $\mathbb{Z}$. When $0 < c < p/2$ this solution is $D = 2c + p$, and when $p/2 < c < p$ this solution is $D = 2c - p$. This matches our result in Proposition 6.2.9.
When $m = 1$, (5.1.4) means that we are looking for smallest odd positive integer D which satisfies $D = -2c$ in $\mathbb{F}_p$, in other words the smallest odd positive integer D which satisfies $D \equiv -2c \bmod p$ in $\mathbb{Z}$. When $0 < c < p/2$ this solution is $D = -2c + p$, and when $p/2 < c < p$ this solution is $D = -2c + 3p$. This matches our result in Proposition 6.2.13.

As in the previous proposition, the irreducible representations classified here are those in category $\mathcal{O}$. This is evident from the condition $x^{2p} = y^{2p} = 0$ in each $W_{1,m}$. The pth power of the S_2-invariant $x^2 \in S(\mathfrak{h}^*)^{S_2}$ is annhilated when moving to the quotient baby Verma module $N_{1,c}(\tau)$, and irreducibles in category $\mathcal{O}$ are baby Verma modules or quotients thereof. Furthermore, the dimension of $N_{1,c}(\tau)$ is $2p$ therefore $y^{2p} = 0$ because the modules in category $\mathcal{O}$ are $\mathbb{Z}$-graded and y lowers the degree by 1.

5.2 *Representations of Cherednik Algebras Associated to Symmetric and Dihedral Groups in Positive Characteristic*, Carl Lian. Preprint on arXiv since 2012 [Li14]

In this preprint, Carl Lian considers the rational Cherednik algebras associated to symmetric and dihedral groups in positive characteristic. In Section 3 of this preprint, Lian considers the algebra $H_{1,c}(S_3, V)$ in characteristic $p > 3$ where $c \in \mathbb{F}_p$. Like us, Lian uses the notation $\mathfrak{h}$ for the reflection representation of S_n but in Lian's work this is the permutation representation which we denote V. Therefore we must be mindful of any discrepancies arising in formulas from the factor of $\left(\frac{1-z^p}{1-z}\right)$. For more detail on this correspondence see Section 4.2.

Lian describes the irreducible modules $L_{1,c}(\texttt{triv})$ arising from the trivial representation $\tau = \texttt{triv} \in \widehat{S_n}$.

Proposition 5.2.1 ([Li14], Theorem 2.8). *If c is not in $\mathbb{F}_p$ then c is a generic value.*

This proposition implies that $c \in \mathbb{F}_p$ are the only values for which c can possibly be special, since all c outside of $\mathbb{F}_p$ behave generically. Note that the converse does not hold since there are cases where $c \in \mathbb{F}_p$ behaves the same as a generic value, as shown in Theorem 9.0.1 where all c behave generically.

Proposition 5.2.2 ([Li14], Theorem 3.1). *When $c = 0$, the Hilbert polynomial of $L_{1,0}(S_n, V, \texttt{triv})$*

is

$$Hilb_{L_{1,0}(S_n,V,\mathtt{triv})}(z) = \left(\frac{1-z^p}{1-z}\right)^n$$

for all n.

To get the Hilbert polynomial of the irreducible module for the rational Cherednik algebra $H_{1,0}(S_n, \mathfrak{h})$ we divide by a factor of $\left(\frac{1-z^p}{1-z}\right)$ and obtain

$$\mathrm{Hilb}_{L_{1,0}(S_n,\mathfrak{h},\mathtt{triv})}(z) = \left(\frac{1-z^p}{1-z}\right)^{n-1}$$

which agrees with our result in Proposition 2.6.13.

Proposition 5.2.3 ([Li14], Theorem 3.2)**.** *When $c = n^{-1}$, the Hilbert polynomial of $L_{1,n^{-1}}(S_n, V, \mathtt{triv})$ is*

$$Hilb_{L_{1,n^{-1}}(S_n,V,\mathtt{triv})}(z) = \frac{1-z^p}{1-z}.$$

As Lian considers $p > 3$ we have $p \nmid n$ and therefore n^{-1} is permissible. After dividing by a factor of $\left(\frac{1-z^p}{1-z}\right)$ the result reads as

$$\mathrm{Hilb}_{L_{1,n^{-1}}(S_n,\mathfrak{h},\mathtt{triv})}(z) = 1$$

which is a special case of Proposition 4.1.4.

In Lian's preprint, the following proposition has an incomplete proof as it relies on work that is unpublished as of yet. There is also an error in the statement of the proposition which we have corrected.

Proposition 5.2.4 ([Li14], Theorem 3.3)**.** *When $n = 3$ and $p > 3$, express c as a positive integer with $c < p$. In the following three cases, the singular vectors which generate the maximal graded submodule of $M_{1,c}(S_3, V, \mathtt{triv})$ are found in the stated degrees.*

- *For $0 < c < p/3$: $p, p + 3c, p + 3c$*
- *For $p/3 < c < p/2$: $3c - p, 3c - p, p$*
- *For $2p/3 < c < p$: $3c - 2p, 3c - 2p, p$*

The following conjecture of Lian is proved by us in Section 12.1.

Conjecture 5.2.5 ([Li14], Remark 3.5)**.** *We conjecture that when $p/2 < c < 2p/3$, the singular vectors which generate the maximal graded submodule of $M_{1,c}(S_3, V, \mathtt{triv})$ are in degrees $6c - 3p, p,$ and p.*

5.3 *Representations of rational Cherednik algebras of $G(m,r,n)$ in positive characteristic*, Sheela Devadas and Steven Sam. Journal of Commutative Algebra (2014) [DeSa14]

In this paper, Sheela Devadas and Steven Sam consider rational Cherednik algebras associated to complex reflection groups $G(m,r,n)$ in positive characteristic, and their representation theory for generic values of the parameter c. In Section 4 of this paper, Devadas and Sam provide characters for $G = G(m,1,n)$ in the non-modular case. This coincides with our work in the case of $G = G(1,1,n)$, which is the symmetric group S_n, in characteristic $p > n$; hence we shall fix $m = r = 1$. The following results stated in the case of $t = 1$ are particular to the rational Cherednik algebra $H_{1,c}(S_n, V)$ and therefore we must consider a factor of $\left(\dfrac{1-z^p}{1-z}\right)$ in order to match their results with ours.

The authors denote by $\underline{\lambda}$ a multi-partition of n, $(\lambda^0, \lambda^1, \dots, \lambda^{m-1})$. Since we fix $m = 1$ we shall interpret $\underline{\lambda}$ as a *partition* of n and write λ instead. For a box s in the Young diagram of λ let hook(s) denote its hook length and let

$$H_\lambda(z) = \prod_{s\in\lambda}(1 - z^{\text{hook}(s)}).$$

Let S_λ denote the irreducible representation of S_n parametrised by λ and χ_λ its character. Let λ^* denote the partition corresponding to the dual representation $S_\lambda{}^*$. Let C_λ be the conjugacy class of S_n containing those permutations of cycle type λ. Let $|Z_\lambda| = |S_n|/|C_\lambda|$ denote the size of the stabilizer subgroup of any element in C_λ under conjugation by S_n, which the authors denote z_λ. For each partition ρ, the authors denote by $\chi_\lambda(\rho)$ the value $\chi_\lambda(g)$ for any $g \in C_\rho$. The authors denote by $\xi \in \Bbbk$ a primitive mth root of unity, hence we fix $\xi = 1$. For partitions μ and λ, the authors define

$$K'_{\mu,\lambda}(z) = H_\lambda(z) \sum_\rho \frac{\chi_\lambda(\rho)\chi_{\mu^*}(\rho)}{|Z_\rho| \prod_j (1 - z^{\rho_j})}$$

where $\rho = (\rho_0, \rho_1, \dots, \rho_k)$ is a partition of n.

Proposition 5.3.1 ([DeSa14], Proposition 4.1)**.** *Consider p not dividing $n!$ and $\tau = S_\lambda$. The character of $L_{0,c}(\tau)$ for generic c is*

$$\chi_{L_{0,c}(\tau)}(z) = \sum_\mu K'_{\mu,\lambda}(z)[S_\mu].$$

In particular, the Hilbert polynomial is

$$Hilb_{L_{0,c}(\tau)}(z) = (dim\ \tau)\frac{H_\lambda(z)}{(1-z)^n}.$$

Let A denote the polynomial algebra $S(V^*) = \Bbbk[x_1, \ldots, x_n]$ and let $A^{(p)} = \Bbbk[x_1^p, \ldots, x_n^p]$. Let $Q = A/A^{(p)}$ and let $[Q]$ be the character $\chi_Q(z)$. We have corresponded with the authors to confirm that there is a typo in their paper. The character formula for $t = 1$ should be written with $K'_{\mu,\lambda}(z^p)$ rather than $K'_{\mu,\lambda}(z)$ and we correct this in the following proposition.

Proposition 5.3.2 ([DeSa14], Proposition 4.2)**.** *Consider p not dividing $n!$ and $\tau = S_\lambda$. The character of $L_{1,c}(S_n, V, \tau)$ for generic c is*

$$\chi_{L_{1,c}(S_n,V,\tau)}(z) = [Q] \sum_{\mu} K'_{\mu,\lambda}(z^p)[S_\mu].$$

In particular, the Hilbert polynomial is

$$Hilb_{L_{1,c}(S_n,V,\tau)}(z) = (dim\ \tau) \frac{H_\lambda(z^p)}{(1-z)^n}.$$

In their proof of this proposition, Devadas and Sam state the following:

> *Ignoring the grading, one has a G-equivariant isomorphism*
>
> $$Q \otimes \Bbbk[S_n] \cong L(S_\lambda)$$
>
> *for all λ.*

The authors denote by $L(S_\lambda)$ the irreducible module $L_{1,c}(S_n, V, \tau)$ where $\tau = S_\lambda$. A basis of Q is given by $\{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i < p\}$ hence $\dim Q = p^n$. Therefore we see that $L(S_\lambda)$ has constant dimension, with

$$\dim\ L(S_\lambda) = \dim\ (Q \otimes \Bbbk[S_n]) = p^n \cdot n!$$

for all λ. Considering the relation

$$\chi_{L_{1,c}(S_n,V,\tau)}(z) = \chi_{L_{1,c}(S_n,V,\tau)}(z) \cdot \left(\frac{1-z^p}{1-z} \right)$$

we get $\dim\ L_{1,c}(S_n, \mathfrak{h}, \tau) = \frac{1}{p} \cdot \dim\ L_{1,c}(S_n, V, \tau) = p^{n-1} \cdot n!$ for all τ.

In order to write the character and Hilbert polynomial of the irreducible modules $L_{1,c}(S_n, \mathfrak{h}, \tau)$ we must modify the formulas given in Proposition 5.3.2. In characteristic $p > n$ considered here, we have $V^* = \mathfrak{h}^* \oplus \Bbbk X$. Therefore $S(V^*)$, which the authors denoted A, decomposes as a tensor product $S(V^*) \cong S(\mathfrak{h}^*) \otimes \Bbbk[X]$. We denote by $S^{(p)}(V^*)$ the quotient

$$S^{(p)}(V^*) = \frac{S(V^*)}{(x^p \mid x \in S(V^*))}$$

and similarly let $S^{(p)}(\mathfrak{h}^*)$ be the quotient

$$S^{(p)}(\mathfrak{h}^*) = \frac{S(\mathfrak{h}^*)}{(x^p \mid x \in S(\mathfrak{h}^*))}.$$

Now we can write

$$S^{(p)}(V^*) = S^{(p)}(\mathfrak{h}^*) \otimes \left(\frac{\Bbbk[X]}{(X^p)}\right).$$

The character of $\Bbbk[X]/(X^p)$ is

$$\left(\frac{1-z^p}{1-z}\right)[\texttt{triv}]$$

therefore, noting that $\chi_{S^{(p)}(V^*)}(z) = [Q]$, we get

$$[Q] = \left(\frac{1-z^p}{1-z}\right)\chi_{S^{(p)}(\mathfrak{h}^*)}(z).$$

Hence our modified formula for the irreducible characters is

$$\begin{aligned}
\chi_{L_{1,c}(S_n,\mathfrak{h},\tau)}(z) &= \left(\frac{1-z}{1-z^p}\right)\chi_{L_{1,c}(S_n,V,\tau)}(z) \\
&= \left(\frac{1-z}{1-z^p}\right)[Q]\sum_{\mu} K'_{\mu,\lambda}(z^p)[S_\mu] \\
&= \left(\frac{1-z}{1-z^p}\right)\left(\frac{1-z^p}{1-z}\right)\chi_{S^{(p)}(\mathfrak{h}^*)}\sum_{\mu} K'_{\mu,\lambda}(z^p)[S_\mu] \\
&= \chi_{S^{(p)}(\mathfrak{h}^*)}(z)\sum_{\mu} K'_{\mu,\lambda}(z^p)[S_\mu].
\end{aligned}$$

To calculate the Hilbert polynomial of the irreducible module $L_{1,c}(S_n,\mathfrak{h},\tau)$ we shall use the formula in Proposition 5.3.2 and multiply by a factor of $\left(\frac{1-z}{1-z^p}\right)$ to get

$$\begin{aligned}
\mathrm{Hilb}_{L_{1,c}(S_n,\mathfrak{h},\tau)}(z) &= \mathrm{Hilb}_{L_{1,c}(S_n,\mathfrak{h},\tau)}(z)\cdot\left(\frac{1-z}{1-z^p}\right) \\
&= (\dim\tau)\frac{H_\lambda(z)}{(1-z)^n}\cdot\left(\frac{1-z}{1-z^p}\right) \\
&= (\dim\tau)\frac{H_\lambda(z)}{(1-z)^{n-1}(1-z^p)}.
\end{aligned}$$

We will now calculate Hilbert polynomials and characters of irreducible modules using these formulas and show that they agree with our results.

When $n=2$ we have $S_2 = \{e,(12)\}$ and consider $p \geq 3$. The partitions of 2 are $\lambda_1 = (2)$ and $\lambda_2 = (1,1)$. These correspond to irreducible representations $S_{\lambda_1} = \texttt{triv}$, $S_{\lambda_2} = \texttt{sign}$, and conjugacy classes $C_{\lambda_1} = \{(12)\}$, $C_{\lambda_2} = \{e\}$. We have $H_{\lambda_1} = H_{\lambda_2} = (1-z^2)(1-z)$.

Therefore when $t = 0$ we have

$$\mathrm{Hilb}_{L_{0,c}(\mathtt{triv})}(z) = \frac{H_{\lambda_1}(z)}{(1-z)^2} = \frac{1-z^2}{1-z} = 1+z$$

and

$$\mathrm{Hilb}_{L_{0,c}(\mathtt{sign})}(z) = \frac{H_{\lambda_2}(z)}{(1-z)^2} = 1+z.$$

This matches our results in Corollaries 6.1.6 and 6.1.8. To calculate the characters we have $|Z_{\lambda_1}| = |Z_{\lambda_2}| = 2$. Hence

$$\begin{aligned} K'_{\lambda_1,\lambda_1}(z) &= H_{\lambda_1}(z)\left(\frac{\chi_{\mathtt{triv}}(\lambda_1)\chi_{\mathtt{triv}^*}(\lambda_1)}{|Z_{\lambda_1}|(1-z^2)} + \frac{\chi_{\mathtt{triv}}(\lambda_2)\chi_{\mathtt{triv}^*}(\lambda_2)}{|Z_{\lambda_2}|(1-z)^2}\right) \\ &= (1-z^2)(1-z)\left(\frac{1}{2(1-z^2)} + \frac{1}{2(1-z)^2}\right) = 1. \end{aligned}$$

Likewise,

$$\begin{aligned} K'_{\lambda_2,\lambda_1}(z) &= H_{\lambda_1}(z)\left(\frac{\chi_{\mathtt{triv}}(\lambda_1)\chi_{\mathtt{sign}^*}(\lambda_1)}{|Z_{\lambda_1}|(1-z^2)} + \frac{\chi_{\mathtt{triv}}(\lambda_2)\chi_{\mathtt{sign}^*}(\lambda_2)}{|Z_{\lambda_2}|(1-z)^2}\right) \\ &= (1-z^2)(1-z)\left(\frac{-1}{2(1-z^2)} + \frac{1}{2(1-z)^2}\right) = z. \end{aligned}$$

From here we can write

$$\chi_{L_{0,c}(\mathtt{triv})}(z) = K'_{\lambda_1,\lambda_1}(z)[S_{\lambda_1}] + K'_{\lambda_2,\lambda_1}(z)[S_{\lambda_2}] = [\mathtt{triv}] + [\mathtt{sign}]z$$

which matches our result in Proposition 6.1.5. Similarly one can calculate

$$\chi_{L_{0,c}(\mathtt{sign})}(z) = K'_{\lambda_1,\lambda_2}(z)[S_{\lambda_1}] + K'_{\lambda_2,\lambda_2}(z)[S_{\lambda_2}] = [\mathtt{sign}] + [\mathtt{triv}]z$$

which matches our result in Proposition 6.1.7.
Note that this follows by Corollary 2.8.3 since $-c$ is a generic value if and only if c is generic.

When $t = 1$, we can calculate the Hilbert polynomial of $L_{1,c}(S_2, \mathfrak{h}, \mathtt{triv})$ as follows.

$$\begin{aligned} \mathrm{Hilb}_{L_{1,c}(S_2,\mathfrak{h},\mathtt{triv})}(z) &= (\dim\tau)\frac{H_{\lambda_1}(z^p)}{(1-z)^{n-1}(1-z^p)} \\ &= \frac{(1-z^{2p})(1-z^p)}{(1-z)(1-z^p)} \\ &= 1+z+z^2+\cdots+z^{2p-1} \end{aligned}$$

and this matches our result in Corollary 6.2.8. Since `triv` and `sign` have the same dimension and $H_{\lambda_1}(z^p) = H_{\lambda_2}(z^p)$, we also get

$$\mathrm{Hilb}_{L_{1,c}(S_2,\mathfrak{h},\mathtt{sign})}(z) = 1+z+z^2+\cdots+z^{2p-1}$$

which agrees with Corollary 6.2.12.
To calculate characters we use

$$\chi_{L_{1,c}(S_2,\mathfrak{h},\tau)}(z) = \chi_{S^{(p)}(\mathfrak{h}^*)}(z) \sum_{\mu} K'_{\mu,\lambda}(z^p)[S_\mu].$$

where τ is the representation parametrised by the partition λ. We must therefore calculate $\chi_{S^{(p)}(\mathfrak{h}^*)}$. When $n = 2$, we have $V^* = \mathrm{span}_{\Bbbk}\{x_1, x_2\}$ and the quotient $\mathfrak{h}^* = V^*/(x_1 + x_2)$ is a $\texttt{sign}$ representation of S_2 with basis $\{\overline{x_1}\}$. Hence the algebra $S^{(p)}(\mathfrak{h}^*)$ has a basis $\{1, \overline{x_1}, \overline{x_1}^2, \ldots, \overline{x_1}^{p-1}\}$ and therefore its character is

$$\begin{aligned}\chi_{S^{(p)}(\mathfrak{h}^*)}(z) &= [\texttt{triv}] + [\texttt{sign}]z + [\texttt{triv}]z^2 + \cdots + [\texttt{triv}]z^{p-1} \\ &= [\texttt{triv}]\frac{1 - z^{p+1}}{1 - z^2} + [\texttt{sign}]\frac{z(1 - z^{p-1})}{1 - z^2}\end{aligned}$$

which is $[\texttt{triv}]$ in even degree and $[\texttt{sign}]$ in odd degree. We previously calculated $K'_{\lambda_1,\lambda_1}(z) = 1$ therefore $K'_{\lambda_1,\lambda_1}(z^p) = 1$. Similarly $K'_{\lambda_2,\lambda_1}(z) = z$ so $K'_{\lambda_2,\lambda_1}(z^p) = z^p$. Hence

$$\begin{aligned}\chi_{L_{1,c}(S_2,\mathfrak{h},\texttt{triv})}(z) &= \chi_{S^{(p)}(\mathfrak{h}^*)} \sum_{\mu} K'_{\mu,\lambda_1}(z^p)[S_\mu] \\ &= \left([\texttt{triv}]\frac{1 - z^{p+1}}{1 - z^2} + [\texttt{sign}]\frac{z(1 - z^{p-1})}{1 - z^2}\right)\left([\texttt{triv}] + [\texttt{sign}]z^p\right) \\ &= [\texttt{triv}]\frac{1 - z^{p+1}}{1 - z^2} + [\texttt{sign}]\frac{z(1 - z^{p-1})}{1 - z^2} + [\texttt{sign}]\frac{z^p(1 - z^{p+1})}{1 - z^2} + [\texttt{triv}]\frac{z^{p+1}(1 - z^{p-1})}{1 - z^2} \\ &= [\texttt{triv}]\frac{1 - z^{2p}}{1 - z^2} + [\texttt{sign}]\frac{z(1 - z^{2p})}{1 - z^2} \\ &= [\texttt{triv}] + [\texttt{sign}]z + [\texttt{triv}]z^2 + \cdots + [\texttt{sign}]z^{2p-1}\end{aligned}$$

which describes a module of dimension $2p$ that spans the representation $[\texttt{triv}]$ in even degree and $[\texttt{sign}]$ in odd degree. This matches our result in Proposition 6.2.7.
Finally we calculate the character of $L_{1,c}(S_2, \mathfrak{h}, \texttt{sign})$. We have $K'_{\lambda_1,\lambda_2}(z) = z$ therefore $K'_{\lambda_1,\lambda_2}(z^p) = z^p$. Similarly $K'_{\lambda_2,\lambda_2}(z) = 1$ so $K'_{\lambda_2,\lambda_2}(z^p) = 1$. Hence

$$\begin{aligned}\chi_{L_{1,c}(S_2,\mathfrak{h},\texttt{sign})}(z) &= \chi_{S^{(p)}(\mathfrak{h}^*)} \sum_{\mu} K'_{\mu,\lambda_2}(z^p)[S_\mu] \\ &= \left([\texttt{triv}]\frac{1 - z^{p+1}}{1 - z^2} + [\texttt{sign}]\frac{z(1 - z^{p-1})}{1 - z^2}\right)\left([\texttt{sign}] + [\texttt{triv}]z^p\right) \\ &= [\texttt{sign}]\frac{1 - z^{p+1}}{1 - z^2} + [\texttt{triv}]\frac{z(1 - z^{p-1})}{1 - z^2} + [\texttt{triv}]\frac{z^p(1 - z^{p+1})}{1 - z^2} + [\texttt{sign}]\frac{z^{p+1}(1 - z^{p-1})}{1 - z^2} \\ &= [\texttt{sign}]\frac{1 - z^{2p}}{1 - z^2} + [\texttt{triv}]\frac{z(1 - z^{2p})}{1 - z^2} \\ &= [\texttt{sign}] + [\texttt{triv}]z + [\texttt{sign}]z^2 + \cdots + [\texttt{triv}]z^{2p-1}\end{aligned}$$

which describes a module of dimension $2p$ that spans [$\texttt{sign}$] in even degree and [$\texttt{triv}$] in odd degree. This matches our result in Proposition 6.2.11.

When $n = 3$ we have $S_3 = \{e, (12), (13), (23), (123), (132)\}$ and consider $p \geq 5$. We have partitions $\lambda_1 = (3)$, $\lambda_2 = (2,1)$ and $\lambda_3 = (1,1,1)$. These correspond to representations $S_{\lambda_1} = \texttt{triv}$, $S_{\lambda_2} = \texttt{stand}$, $S_{\lambda_3} = \texttt{sign}$ and conjugacy classes $C_{\lambda_1} = \{(123), (132)\}$, $C_{\lambda_2} = \{(12), (13), (23)\}$ and $C_{\lambda_3} = \{e\}$. We have $H_{\lambda_1}(z) = H_{\lambda_3}(z) = (1-z^3)(1-z^2)(1-z)$ and $H_{\lambda_2}(z) = (1-z^3)(1-z)^2$. Additionally, $|Z_{\lambda_1}| = 3$, $|Z_{\lambda_2}| = 2$ and $|Z_{\lambda_3}| = 6$.
When $t = 0$ we can calculate the Hilbert polynomial as

$$\begin{aligned}
\mathrm{Hilb}_{L_{0,c}(\texttt{triv})}(z) = \mathrm{Hilb}_{L_{0,c}(\texttt{sign})}(z) &= \frac{H_{\lambda_1}(z)}{(1-z)^3} \\
&= \frac{(1-z^3)(1-z^2)(1-z)}{(1-z)^3} \\
&= (1+z+z^2)(1+z) \\
&= 1 + 2z + 2z^2 + z^3
\end{aligned}$$

which is equal to the Hilbert polynomial of the baby Verma modules $N_{0,c}(\texttt{triv})$ and $N_{0,c}(\texttt{sign})$ as shown in Example 4.1.1. Since the irreducible module $L_{0,c}(\tau)$ is a quotient of the baby Verma module $N_{0,c}(\tau)$, we can therefore conclude that $L_{0,c}(\texttt{triv}) = N_{0,c}(\texttt{triv})$ and $L_{0,c}(\texttt{sign})$ $= N_{0,c}(\texttt{sign})$ as described in Theorem 11.0.1. Similarly the Hilbert polynomial of $L_{0,c}(\texttt{stand})(z)$ is

$$\begin{aligned}
\mathrm{Hilb}_{L_{0,c}(\texttt{stand})}(z) &= 2\frac{H_{\lambda_2}(z)}{(1-z)^3} \\
&= \frac{2(1-z^3)(1-z)^2}{(1-z)^3} \\
&= 2(1+z+z^2) = 2 + 2z + 2z^2
\end{aligned}$$

which shows that $L_{0,c}(\texttt{stand})(z)$ is a proper quotient of $N_{0,c}(\texttt{stand})(z)$.
To calculate the characters, we need the values of $K'_{\mu,\lambda}(z)$ for each pair of partitions, μ and λ. Firstly,

$$\begin{aligned}
K'_{\lambda_1,\lambda_1}(z) &= H_{\lambda_1}(z)\left(\frac{\chi_{\texttt{triv}}(\lambda_1)\chi_{\texttt{triv}^*}(\lambda_1)}{|Z_{\lambda_1}|(1-z^3)} + \frac{\chi_{\texttt{triv}}(\lambda_2)\chi_{\texttt{triv}^*}(\lambda_2)}{|Z_{\lambda_2}|(1-z^2)(1-z)} + \frac{\chi_{\texttt{triv}}(\lambda_3)\chi_{\texttt{triv}^*}(\lambda_3)}{|Z_{\lambda_3}|(1-z)^3}\right) \\
&= (1-z^3)(1-z^2)(1-z)\left(\frac{1}{3(1-z^3)} + \frac{1}{2(1-z^2)(1-z)} + \frac{1}{6(1-z)^3}\right) \\
&= \frac{1}{6}\bigg(2(1-z^2)(1-z) + 3(1-z^3) + (1+z+z^2)(1+z)\bigg) = 1.
\end{aligned}$$

Likewise,

$$\begin{aligned}K'_{\lambda_2,\lambda_1}(z) &= H_{\lambda_1}(z)\Big(\frac{\chi_{\texttt{triv}}(\lambda_1)\chi_{\texttt{stand}^*}(\lambda_1)}{|Z_{\lambda_1}|(1-z^3)} + \frac{\chi_{\texttt{triv}}(\lambda_2)\chi_{\texttt{stand}^*}(\lambda_2)}{|Z_{\lambda_2}|(1-z^2)(1-z)} + \frac{\chi_{\texttt{triv}}(\lambda_3)\chi_{\texttt{stand}^*}(\lambda_3)}{|Z_{\lambda_3}|(1-z)^3}\Big)\\ &= (1-z^3)(1-z^2)(1-z)\Big(\frac{-1}{3(1-z^3)} + \frac{0}{2(1-z^2)(1-z)} + \frac{2}{6(1-z)^3}\Big)\\ &= \frac{1}{3}\Big(-(1-z^2)(1-z) + (1+z+z^2)(1+z)\Big) = z + z^2\end{aligned}$$

Finally,

$$\begin{aligned}K'_{\lambda_3,\lambda_1}(z) &= H_{\lambda_1}(z)\Big(\frac{\chi_{\texttt{triv}}(\lambda_1)\chi_{\texttt{sign}^*}(\lambda_1)}{|Z_{\lambda_1}|(1-z^3)} + \frac{\chi_{\texttt{triv}}(\lambda_2)\chi_{\texttt{sign}^*}(\lambda_2)}{|Z_{\lambda_2}|(1-z^2)(1-z)} + \frac{\chi_{\texttt{triv}}(\lambda_3)\chi_{\texttt{sign}^*}(\lambda_3)}{|Z_{\lambda_3}|(1-z)^3}\Big)\\ &= (1-z^3)(1-z^2)(1-z)\Big(\frac{1}{3(1-z^3)} + \frac{-1}{2(1-z^2)(1-z)} + \frac{1}{6(1-z)^3}\Big)\\ &= \frac{1}{6}\Big(2(1-z^2)(1-z) - 3(1-z^3) + (1+z+z^2)(1+z)\Big) = z^3.\end{aligned}$$

Therefore the character of $L_{0,c}(S_3, \mathfrak{h}, \texttt{triv})$ is

$$\begin{aligned}\chi_{L_{0,c}(\texttt{triv})}(z) &= K'_{\lambda_1,\lambda_1}(z)[S_{\lambda_1}] + K'_{\lambda_2,\lambda_1}(z)[S_{\lambda_2}] + K'_{\lambda_3,\lambda_1}(z)[S_{\lambda_3}]\\ &= [\texttt{triv}] + [\texttt{stand}]z + [\texttt{stand}]z^2 + [\texttt{sign}]z^3.\end{aligned}$$

Similarly we can compute $K'_{\lambda_1,\lambda_2}(z) = z$, $K'_{\lambda_2,\lambda_2}(z) = 1 + z^2$ and $K'_{\lambda_3,\lambda_2}(z) = z$, hence the character of $L_{0,c}(S_3, \mathfrak{h}, \texttt{stand})$ is

$$\begin{aligned}\chi_{L_{0,c}(\texttt{stand})}(z) &= K'_{\lambda_1,\lambda_2}(z)[S_{\lambda_1}] + K'_{\lambda_2,\lambda_2}(z)[S_{\lambda_2}] + K'_{\lambda_3,\lambda_2}(z)[S_{\lambda_3}]\\ &= [\texttt{stand}] + ([\texttt{triv}] + [\texttt{sign}])z + [\texttt{stand}]z^2\end{aligned}$$

which agrees the character of the module we calculate in Lemma 11.1.1, therefore that module we calculate is irreducible. Lastly, $K'_{\lambda_1,\lambda_3}(z) = z^3$, $K'_{\lambda_2,\lambda_3}(z) = z + z^2$, $K'_{\lambda_3,\lambda_3}(z) = 1$ and

$$\begin{aligned}\chi_{L_{0,c}(\texttt{sign})}(z) &= K'_{\lambda_1,\lambda_3}(z)[S_{\lambda_1}] + K'_{\lambda_2,\lambda_3}(z)[S_{\lambda_2}] + K'_{\lambda_3,\lambda_3}(z)[S_{\lambda_3}]\\ &= [\texttt{sign}] + [\texttt{stand}]z + [\texttt{stand}]z^2 + [\texttt{triv}]z^3.\end{aligned}$$

When $t = 1$ we calculate the Hilbert polynomials of irreducible modules using the formula

$$\mathrm{Hilb}_{L_{1,c}(S_3,\mathfrak{h},\tau)}(z) = (\dim\,\tau)\frac{H_\lambda(z^p)}{(1-z)^2(1-z^p)}.$$

Therefore

$$\mathrm{Hilb}_{L_{1,c}(S_3,\mathfrak{h},\texttt{triv})}(z) = \frac{(1-z^{3p})(1-z^{2p})(1-z^p)}{(1-z)^2(1-z^p)} =$$

$$= (1 + z + z^2 + \cdots + z^{3p-1})(1 + z + z^2 + \cdots + z^{2p-1})$$

and this is the Hilbert polynomial of the baby Verma module $N_{1,c}(S_3, \mathfrak{h}, \texttt{triv})$ as shown in Example 4.1.2. Since the irreducible module $L_{1,c}(S_3, \mathfrak{h}, \texttt{triv})$ is a quotient of the baby Verma module $N_{1,c}(S_3, \mathfrak{h}, \texttt{triv})$, we can conclude that $L_{1,c}(S_3, \mathfrak{h}, \texttt{triv}) = N_{1,c}(S_3, \mathfrak{h}, \texttt{triv})$ as they have the same Hilbert polynomial. This result is found in Theorem 11.0.1. Equally,

$$\begin{aligned}\mathrm{Hilb}_{L_{1,c}(S_3,\mathfrak{h},\texttt{sign})}(z) &= \frac{(1-z^{3p})(1-z^{2p})(1-z^p)}{(1-z)^2(1-z^p)} \\ &= (1 + z + z^2 + \cdots + z^{3p-1})(1 + z + z^2 + \cdots + z^{2p-1})\end{aligned}$$

which is the Hilbert polynomial of $N_{1,c}(S_3, \mathfrak{h}, \texttt{sign})$, so by the same argument $L_{1,c}(S_3, \mathfrak{h}, \texttt{sign})$ $= N_{1,c}(S_3, \mathfrak{h}, \texttt{sign})$ and this result appears in Theorem 11.0.1. The Hilbert polynomial of $L_{1,c}(S_3, \mathfrak{h}, \texttt{stand})$ is

$$\begin{aligned}\mathrm{Hilb}_{L_{1,c}(S_3,\mathfrak{h},\texttt{stand})}(z) &= 2\frac{(1-z^{3p})(1-z^p)^2}{(1-z)^2(1-z^p)} \\ &= 2(1 + z + z^2 + \cdots + z^{3p-1})(1 + z + z^2 + \cdots + z^{p-1}).\end{aligned}$$

this agrees with the Hilbert polynomial of a module we calculate in Lemma 11.2.19, therefore that module is irreducible. To calculate characters, we will use the formula

$$\chi_{L_{1,c}(S_3,\mathfrak{h},\tau)}(z) = \chi_{S^{(p)}(\mathfrak{h}^*)}(z) \sum_{\mu} K'_{\mu,\lambda}(z^p)[S_\mu].$$

As we have previously calculated the values of $K'_{\mu,\lambda}(z)$ we can write the values of $K'_{\mu,\lambda}(z^p)$ as follows.

$$\begin{array}{lll} K_{\lambda_1,\lambda_1}(z^p) = 1 & K_{\lambda_2,\lambda_1}(z^p) = z^p + z^{2p} & K_{\lambda_3,\lambda_1}(z^p) = z^{3p} \\ K_{\lambda_1,\lambda_2}(z^p) = z^p & K_{\lambda_2,\lambda_2}(z^p) = 1 + z^{2p} & K_{\lambda_3,\lambda_2}(z^p) = z^p \\ K_{\lambda_1,\lambda_3}(z^p) = z^{3p} & K_{\lambda_2,\lambda_3}(z^p) = z^p + z^{2p} & K_{\lambda_3,\lambda_3}(z^p) = 1 \end{array}$$

We also require the character of $S^{(p)}(\mathfrak{h}^*)$ which is

$$\chi_{S^{(p)}(\mathfrak{h}^*)}(z) = \chi_{S(\mathfrak{h}^*)}(z) \cdot (1 - [\texttt{stand}]z^p + [\texttt{sign}]z^{2p})$$

as proven in Corollary 7.2.6. Now

$$\begin{aligned}\chi_{L_{1,c}(S_3,\mathfrak{h},\texttt{triv})}(z) &= \chi_{S^{(p)}(\mathfrak{h}^*)}(z)\Big(K_{\lambda_1,\lambda_1}(z^p)[S_{\lambda_1}] + K_{\lambda_2,\lambda_1}(z^p)[S_{\lambda_2}] + K_{\lambda_3,\lambda_1}(z^p)[S_{\lambda_3}]\Big) \\ &= \chi_{S^{(p)}(\mathfrak{h}^*)}(z)\big([\texttt{triv}] + [\texttt{stand}](z^p + z^{2p}) + [\texttt{sign}]z^{3p}\big) \\ &= \chi_{S(\mathfrak{h}^*)}(z) \cdot (1 - [\texttt{stand}]z^p + [\texttt{sign}]z^{2p})\big([\texttt{triv}] + [\texttt{stand}](z^p + z^{2p}) + [\texttt{sign}]z^{3p}\big)\end{aligned}$$

$$
\begin{aligned}
&= \chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{triv}] - [\texttt{triv}]z^{2p} - [\texttt{triv}]z^{3p} + [\texttt{triv}]z^{5p}\big) \\
&= \chi_{S(\mathfrak{h}^*)}(z) \cdot (1 - z^{2p})(1 - z^{3p})
\end{aligned}
$$

$$
\begin{aligned}
\chi_{L_{1,c}(S_3,\mathfrak{h},\texttt{stand})}(z) &= \chi_{S^{(p)}(\mathfrak{h}^*)}(z)\Bigg(K_{\lambda_1,\lambda_2}(z^p)[S_{\lambda_1}] + K_{\lambda_2,\lambda_2}(z^p)[S_{\lambda_2}] + K_{\lambda_3,\lambda_2}(z^p)[S_{\lambda_3}]\Bigg) \\
&= \chi_{S^{(p)}(\mathfrak{h}^*)}(z)\big([\texttt{stand}] + ([\texttt{triv}] + [\texttt{sign}])z^p + [\texttt{stand}]z^{2p}\big) \\
&= \chi_{S(\mathfrak{h}^*)}(z) \cdot (1 - [\texttt{stand}]z^p + [\texttt{sign}]z^{2p})\big([\texttt{stand}] + ([\texttt{triv}] + [\texttt{sign}])z^p + [\texttt{stand}]z^{2p}\big) \\
&= \chi_{S(\mathfrak{h}^*)}(z) \cdot ([\texttt{stand}] - [\texttt{stand}]z^p - [\texttt{stand}]z^{3p} + [\texttt{stand}]z^{4p}) \\
&= \chi_{S(\mathfrak{h}^*)}(z) \cdot [\texttt{stand}](1 - z^p)(1 - z^{3p})
\end{aligned}
$$

$$
\begin{aligned}
\chi_{L_{1,c}(S_3,\mathfrak{h},\texttt{sign})}(z) &= \chi_{S^{(p)}(\mathfrak{h}^*)}(z)\Bigg(K_{\lambda_1,\lambda_3}(z^p)[S_{\lambda_1}] + K_{\lambda_2,\lambda_3}(z^p)[S_{\lambda_2}] + K_{\lambda_3,\lambda_3}(z^p)[S_{\lambda_3}]\Bigg) \\
&= \chi_{S^{(p)}(\mathfrak{h}^*)}(z)\big([\texttt{sign}] + [\texttt{stand}](z^p + z^{2p}) + [\texttt{triv}]z^{3p}\big) \\
&= \chi_{S(\mathfrak{h}^*)}(z) \cdot (1 - [\texttt{stand}]z^p + [\texttt{sign}]z^{2p})\big([\texttt{sign}] + [\texttt{stand}](z^p + z^{2p}) + [\texttt{triv}]z^{3p}\big) \\
&= \chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{sign}] - [\texttt{sign}]z^{2p} - [\texttt{sign}]z^{3p} + [\texttt{sign}]z^{5p}\big) \\
&= \chi_{S(\mathfrak{h}^*)}(z) \cdot [\texttt{sign}](1 - z^{2p})(1 - z^{3p})
\end{aligned}
$$

and these results are summarised in Theorem 11.0.1.

5.4 *The polynomial representation of the type A_{n-1} rational Cherednik algebra in characteristic $p \mid n$*, Sheela Devadas and Yi Sun. Communications in Algebra (2016) [DeSu16]

This paper of Sheela Devadas and Yi Sun examines the characteristic p representation theory of rational Cherednik algebras of type A_{n-1} in the case that $p \mid n$ and $\tau = \texttt{triv}$. This is relevant for us in the cases $(n,p) = (2,2)$ and $(n,p) = (3,3)$.

The notation used by the authors matches our notation for the majority of objects except $t = \hbar$ and $H_{t,c}(S_n,\mathfrak{h}) = \mathcal{H}_{\hbar,c}(\mathfrak{h})$. However this paper focuses only on the $t = 1$ case. Additionally, the bars over the elements of $\mathfrak{h}^*$ are omitted so we must identify $x_i = \overline{x_i}$ in their work. The authors denote by A the polynomial algebra $S(\mathfrak{h}^*)$ which is isomorphic to the Verma module $M_{t,c}(S_n,\mathfrak{h},\texttt{triv})$. The authors define a formal power series

$$
F_i(z) = \frac{1}{1 - \overline{x_i}z} \sum_{m=0}^{p-1} \binom{c}{m} \Bigg(\prod_{j=1}^{n} (1 - \overline{x_j}z) - 1 \Bigg)^m
$$

for $i \in \{1,\ldots,n-1\}$ and binomial coefficients $\binom{c}{m} = \frac{c(c-1)\cdots(c-m+1)}{m!}$. For a formal power series $r(z)$ the authors denote by $[z^l]r(z)$ the coefficient of z^l in $r(z)$. Let f_i denote the coefficients of z^p in $F_i(z)$, that is $f_i = [z^p]F_i(z)$ for $i \in \{1, \ldots, n-1\}$.

Theorem 5.4.1 ([DeSu16], Theorem 4.1)**.** *For generic c, the polynomials $f_1, \ldots, f_{n-1}$ are linearly independent and generate the maximal proper graded submodule of $M_{t,c}(S_n, \mathfrak{h}, \texttt{triv})$. The irreducible quotient $L_{1,c}(\texttt{triv}) = M_{1,c}(\texttt{triv})/(f_1, \ldots, f_{n-1})$ is a complete intersection with Hilbert polynomial*

$$Hilb_{L_{1,c}(\texttt{triv})}(z) = \left(\frac{1-z^p}{1-z}\right)^{n-1}.$$

In other words, $f_1, \ldots, f_{n-1}$ are the singular vectors of $M_{1,c}(\texttt{triv})$ and are as linearly independent as possible. Since f_i has degree p for all i, we can conclude that when $p \mid n$ and for generic c, the irreducible module $L_{1,c}(S_n, \mathfrak{h}, \texttt{triv})$ is the quotient of $M_{1,c}(S_n, \mathfrak{h}, \texttt{triv})$ by $n-1$ singular vectors, all of degree p. This means that when $n = p = 2$ there is one singular vector of degree 2, which agrees with our result Proposition 6.2.1. When $n = p = 3$ there are two singular vectors both of degree 3, which agrees with our result Lemma 9.2.1. We will calculate these singular vectors below, noting that we have $\overline{x_1} + \overline{x_2} + \cdots + \overline{x_n} = 0$ in the rational Cherednik algebra and its induced Verma modules.

When $n = p = 2$ there is one power series to consider,

$$\begin{aligned} F_1(z) &= \frac{1}{1-\overline{x_1}z}\left(\binom{c}{0} + \binom{c}{1}\big((1-\overline{x_1}z)(1-\overline{x_2}z)-1\big)\right) \\ &= \frac{1}{1-\overline{x_1}z}\Big(1 + c\big(\overline{x_1x_2}z^2 - (\overline{x_1}+\overline{x_2})z\big)\Big) \\ &= \frac{1}{1-\overline{x_1}z}\big(1 + c\overline{x_1x_2}z^2\big). \end{aligned}$$

The expression $\frac{1}{1-\overline{x_1}z}$ is the geometric series $(1 + \overline{x_1}z + \overline{x_1}^2z^2 + \cdots)$. Therefore

$$\begin{aligned} f_1 = [z^2]F_1(z) &= [z^2]\frac{1}{1-\overline{x_1}z}\big(1 + c\overline{x_1x_2}z^2\big) \\ &= [z^2](1 + \overline{x_1}z + \overline{x_1}^2z^2)(1 + c\overline{x_1x_2}z^2) \\ &= \overline{x_1}^2 + c\overline{x_1x_2}. \end{aligned}$$

Since we have $\overline{x_1} + \overline{x_2} = 0$ and the characteristic is 2, this simplifies as $f_1 = (c+1)\overline{x_1}^2$. This vector is zero when $c = 1$ but for generic values of c the vector is nonzero and agrees with the singular vector we find for generic c in Proposition 6.2.1, up to a nonzero scalar.

When $n = p = 3$ we calculate the following.

$$\begin{aligned} f_1 &= [z^3]F_1(z) \\ &= [z^3]\frac{1}{1-\overline{x_1}z}\left(\binom{c}{0} + \binom{c}{1}\big((1-\overline{x_1}z)(1-\overline{x_2}z)(1-\overline{x_3}z)-1\big)\right. \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \binom{c}{2}\Big((1-\overline{x_1}z)(1-\overline{x_2}z)(1-\overline{x_3}z)-1\Big)^2\Bigg) \\
&= [z^3]\frac{1}{1-\overline{x_1}z}\Bigg(1 + c\big(-(\overline{x_1}+\overline{x_2}+\overline{x_3})z + (\overline{x_1x_2}+\overline{x_1x_3}+\overline{x_2x_3})z^2 - \overline{x_1x_2x_3}z^3\big) \\
&\qquad\qquad + \frac{c(c-1)}{2}\big(-(\overline{x_1}+\overline{x_2}+\overline{x_3})z + (\overline{x_1x_2}+\overline{x_1x_3}+\overline{x_2x_3})z^2 - \overline{x_1x_2x_3}z^3\big)^2\Bigg) \\
&= [z^3](1+\overline{x_1}z+\overline{x_1}^2z^2+\overline{x_1}^3z^3)\big(1+c(\overline{x_1x_2}+\overline{x_1x_3}+\overline{x_2x_3})z^2 - c\overline{x_1x_2x_3}z^3\big) \\
&= \overline{x_1}^3 + c\overline{x_1}(\overline{x_1x_2}+\overline{x_1x_3}+\overline{x_2x_3}) - c\overline{x_1x_2x_3}
\end{aligned}
$$

Making the substitution $\overline{x_3} = -\overline{x_1} - \overline{x_2}$ this simplifies to $f_1 = (1-c)\overline{x_1}^3$ and by symmetry we can also conclude $f_2 = (1-c)\overline{x_2}^3$. Once again, f_1 and f_2 are zero when $c = 1$, but for generic values of c this is not the case and these vectors are scalar multiples of the singular vectors we find for generic c in Lemma 9.2.1.

5.5 *The Hilbert Series of the Irreducible Quotient of the Polynomial Representation of the Rational Cherednik Algebra of Type A_{n-1} in Characteristic p for $p \mid n-1$,* Merrick Cai and Daniil Kalinov. Journal of Algebra and Its Applications (2021) [CaKa21]

In this paper, Merrick Cai and Daniil Kalinov study the rational Cherednik algebra of type A_{n-1}, that is is $H_{t,c}(S_n, \mathfrak{h})$, and its representation theory in characteristic $p \mid n-1$. This only coincides with our work for $(n, p) = (3, 2)$.

Theorem 5.5.1 ([CaKa21], Theorem 2.11)**.** *The Hilbert polynomial of $L_{0,c}(S_n, \mathfrak{h}, \texttt{triv})$ when $p = 2$ is*

$$\mathit{Hilb}_{L_{0,c}(S_n,\mathfrak{h},\texttt{triv})}(z) = (1+z)(1+(n-2)z+z^2).$$

The above theorem shows that the Hilbert polynomial of $L_{0,c}(\texttt{triv})$ for $H_{t,c}(S_3, \mathfrak{h})$ in characteristic 2 is

$$\mathrm{Hilb}_{L_{0,c}(S_3,\mathfrak{h},\texttt{triv})}(z) = (1+z)(1+z+z^2) = 1+2z+2z^2+z^3$$

which matches the result we prove in Lemma 8.1.2.

Theorem 5.5.2 ([CaKa21], Theorem 3.17)**.** *The Hilbert polynomial of $L_{1,c}(S_n, \mathfrak{h}, \texttt{triv})$ when $p = 2$ is*

$$\mathit{Hilb}_{L_{1,c}(S_n,\mathfrak{h},\texttt{triv})}(z) = (1+z^2)(1+z)^{n-1}(1+(n-2)z^2+z^4),$$

or alternatively,

$$\mathit{Hilb}_{L_{1,c}(S_n,\mathfrak{h},\texttt{triv})}(z) = (1+z)^{n-1}(1+(n-1)z^2+(n-1)z^4+z^6.)$$

The above theorem shows that the Hilbert polynomial of $L_{1,c}(\mathtt{triv})$ for $H_{t,c}(S_3, \mathfrak{h})$ in characteristic 2 is

$$\mathrm{Hilb}_{L_{1,c}(S_3,\mathfrak{h},\mathtt{triv})}(z) = (1+z)^2(1+2z^2+2z^4+z^6) = 1+2z+3z^2+4z^3+4z^4+4z^5+3z^6+2z^7+z^8$$

which matches the result we prove in Lemma 8.2.2.

5.6 *Category $\mathcal{O}$ for rational Cherednik algebras $H_{t,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$ in characteristic p*, Martina Balagović and Harrison Chen. Journal of Pure and Applied Algebra (2013) [BaCh13b]

This paper is the sequel to *"Representations of Rational Cherednik Algebras in Positive Characteristic"* ([BaCh13a]), in which Martina Balagović and Harrison Chen examine the representation theory of a rational Cherednik algebra arising from a finite group of Lie type. Although this does not overlap with our work, they do consider a very similar problem and so we thought it worthy of mention.

5.7 *Cherednik algebras and Hilbert schemes in characteristic p*, Roman Bezrukavnikov, Michael Finkelberg and Victor Ginzburg. Representation Theory (2006) [BFG06]

This paper of Roman Bezrukavnikov, Michael Finkelberg and Victor Ginzburg relates to our work as it examines the positive characteristic representation theory of rational Cherednik algebras of type A_{n-1}. However, their approach is geometric and they answer different questions than we do. Their results do appear in more closely related work, for example see [DeSa14], proof of Proposition 4.2.

5.8 *On The Smoothness Of Centres Of Rational Cherednik Algebras In Positive Characteristic*, Gwyn Bellamy and Maurizio Martino. Glasgow Mathematical Journal (2013) [BeMa13]

In their work, Gwyn Bellamy and Maurizio Martino examine the representation theory of restricted rational Cherednik algebras, which are finite-dimensional quotients of rational Cherednik algebras. In particular, they describe the blocks of the restricted rational Cherednik algebra in positive characteristic.

Chapter 6

Irreducible Representations of $H_{t,c}(S_2, \mathfrak{h})$

Our group is $S_2 = \{e, s\}$ where $s = (12)$ is a transposition. We have $V = \text{span}_{\Bbbk}\{y_1, y_2\}$ and $\mathfrak{h} = \text{span}_{\Bbbk}\{y_1 - y_2\}$. Furthermore, $V^* = \text{span}_{\Bbbk}\{x_1, x_2\}$ and the quotient $\mathfrak{h}^* = V^*/(x_1 + x_2)$ is a `sign` representation of S_2 with basis $\{\overline{x_1}\}$.
In characteristic $p \nmid 2$, that is for all $p \geq 3$, we can realise $\mathfrak{h}^*$ as a subrepresentation of V^* through the map $\pi : \mathfrak{h}^* \to V^*$ defined in Proposition 3.2.2. On the basis of $\mathfrak{h}^*$ this map is $\pi(\overline{x_1}) = x_1 - \frac{X}{2} = \frac{1}{2}(x_1 - x_2)$ which gives the basis of $\mathfrak{h}^*$ realised as a subrepresentation of V^*. For notational convenience, let $\{x\}$ be the basis of $\mathfrak{h}^*$ where the variable x is defined by

$$x = \begin{cases} \overline{x_1} & \text{when } p = 2, \\ \frac{1}{2}(x_1 - x_2) & \text{when } p \geq 3. \end{cases}$$

Hence $S(\mathfrak{h}^*)$ is the polynomial algebra in one variable $\Bbbk[x]$. Let $\tau \in \widehat{S_2}$ be an irreducible representation in characteristic p with basis $\{v\}$. The Verma module $M_{t,c}(\tau)$ is isomorphic to $\Bbbk[x] \otimes \tau$ with basis $\{x^i \otimes v \mid i = 0, 1, 2, \dots\}$. The quotient of $M_{t,c}(\tau)$ by any vector $x^k \otimes v$ leaves a finite-dimensional module with basis $\{1 \otimes v, x \otimes v, x^2 \otimes v, \dots, x^{k-1} \otimes v\}$ and everything of degree k and higher is annihilated. Therefore to calculate the irreducible modules $L_{t,c}(\tau)$ we only need to find one singular vector with minimal degree, as any other singular vectors will be contained in a submodule generated by singular vectors of smaller degree.
A singular vector $x^k \otimes v \in M_{t,c}(\tau)$ must satisfy $D_y(x^k \otimes v) = 0$ for all $y \in \mathfrak{h}$; however $\mathfrak{h}$ is 1-dimensional so it suffices to check this for $y = y_1 - y_2$ only. The Dunkl operator is given explicitly by the formula

$$D_{y_1 - y_2}(x^k \otimes v) = t\partial_{y_1 - y_2}(x^k) \otimes v - c\langle \alpha_s, y_1 - y_2 \rangle \frac{x^k - s.x^k}{\alpha_s} \otimes s.v.$$

We have $\alpha_s = \overline{x_1} - \overline{x_2}$ for $s = (12)$, therefore $\alpha_s = 2\overline{x_1}$ because $\overline{x_1} + \overline{x_2} = 0$. An issue arises when considering α_s in characteristic 2, as described in the following remark.

Remark 6.0.1. When $p = 2$, the symmetric group S_2 does not satisfy Definition 1.1.6 of a reflection group. In particular, S_2 is generated by the transposition $s = (12)$ but the rank of $1 - s$ on $\mathfrak{h}$ is 0, not 1. However, we may still carry out the procedure of finding singular vectors with these parameters and, at least when $t = 1$, the results align with the work of Devadas and Sun [DeSu16] on rational Cherednik algebras in characteristic $p \mid n$, as detailed in Section 5.4. We shall still write $\alpha_s = 2x$ because $\langle y_1 - y_2, \alpha_s\rangle = 2$ so this coefficient cancels in the expression for the Dunkl operator.

The partial derivative $\partial_{y_1-y_2}(x^k)$ behaves as an ordinary derivative, since $\langle y_1 - y_2, x\rangle = 1$. We will also write $\chi_\tau(s)v$ for $s.v$. Putting these values into the Dunkl operator, we get

$$\begin{aligned} D_{y_1-y_2}(x^k \otimes v) &= t\partial_{y_1-y_2}(x^k) \otimes v - 2c\frac{x^k - s.x^k}{2x} \otimes \chi_\tau(s)v \\ &= tkx^{k-1} \otimes v - c\chi_\tau(s)(1-(-1)^k)x^{k-1} \otimes v \\ &= \big(tk - (1-(-1)^k)c\chi_\tau(s)\big)x^{k-1} \otimes v \end{aligned} \tag{6.0.2}$$

and we are interested in the smallest positive integer k for which this expression equals 0.

Theorem 6.0.3. *The characters and Hilbert polynomials of the irreducible representation $L_{t,c}(\tau)$ of the rational Cherednik algebra $H_{t,c}(S_2,\mathfrak{h})$, for any p, t, c and τ, are given by the following tables. In all cases, the singular vector is known.*

Characters:

$p = 2$	$\tau = \mathtt{triv}$
$t = 0$, *all* c	$[\mathtt{triv}]$
$t = 1$, *all* c	$[\mathtt{triv}] + [\mathtt{triv}]z$

$\boldsymbol{p > 2}$	$\tau = \texttt{triv}$
$t = 0,\ c = 0$	$[\texttt{triv}]$
$t = 0,\ c \neq 0$	$[\texttt{triv}] + [\texttt{sign}]z$
$t = 1,\ c \notin \mathbb{F}_p$	$[\texttt{triv}]\dfrac{1 - z^{2p}}{1 - z^2} + [\texttt{sign}]\dfrac{z(1 - z^{2p})}{1 - z^2}$
$t = 1,$ $0 \leq c < p/2$	$[\texttt{triv}]\dfrac{1 - z^{2c+p+1}}{1 - z^2} + [\texttt{sign}]\dfrac{z(1 - z^{2c+p-1})}{1 - z^2}$
$t = 1,$ $p/2 < c < p$	$[\texttt{triv}]\dfrac{1 - z^{2c-p+1}}{1 - z^2} + [\texttt{sign}]\dfrac{z(1 - z^{2c-p-1})}{1 - z^2}$

$\boldsymbol{p > 2}$	$\tau = \texttt{sign}$
$t = 0,\ c = 0$	$[\texttt{sign}]$
$t = 0,\ c \neq 0$	$[\texttt{sign}] + [\texttt{triv}]z$
$t = 1,\ c \notin \mathbb{F}_p$	$[\texttt{sign}]\dfrac{1 - z^{2p}}{1 - z^2} + [\texttt{triv}]\dfrac{z(1 - z^{2p})}{1 - z^2}$
$t = 1,$ $0 \leq c < p/2$	$[\texttt{sign}]\dfrac{1 - z^{-2c+p+1}}{1 - z^2} + [\texttt{triv}]\dfrac{z(1 - z^{-2c+p-1})}{1 - z^2}$
$t = 1,$ $p/2 < c < p$	$[\texttt{sign}]\dfrac{1 - z^{-2c+3p+1}}{1 - z^2} + [\texttt{triv}]\dfrac{z(1 - z^{-2c+3p-1})}{1 - z^2}$

Hilbert polynomials:

$\boldsymbol{p = 2}$	$\tau = \texttt{triv}$
$t = 0,$ *all* c	1
$t = 1,$ *all* c	$1 + z$

$\boldsymbol{p > 2}$	$\tau = \texttt{triv}$
$t = 0,\ c = 0$	1
$t = 0,\ c \neq 0$	$1 + z$
$t = 1,\ c \notin \mathbb{F}_p$	$\dfrac{1 - z^{2p}}{1 - z}$
$t = 1,$ $0 \leq c < p/2$	$\dfrac{1 - z^{2c+p}}{1 - z}$
$t = 1,$ $p/2 < c < p$	$\dfrac{1 - z^{2c-p}}{1 - z}$

$\boldsymbol{p > 2}$	$\tau = \texttt{sign}$
$t = 0,\ c = 0$	1
$t = 0,\ c \neq 0$	$1 + z$
$t = 1,\ c \notin \mathbb{F}_p$	$\dfrac{1 - z^{2p}}{1 - z}$
$t = 1,$ $0 \leq c < p/2$	$\dfrac{1 - z^{-2c+p}}{1 - z}$
$t = 1,$ $p/2 < c < p$	$\dfrac{1 - z^{-2c+3p}}{1 - z}$

The remainder of this chapter is a proof of Theorem 6.0.3, divided into sections by case.

6.1 $t = 0$

In this section we describe the irreducible modules for the rational Cherednik algebra $H_{0,c}(S_2, \mathfrak{h})$. When $c = 0$ the irreducible module $L_{0,0}(\tau)$ has character $\chi_{L_{0,0}(\tau)} = [\tau]$ as described in Proposition 2.6.11. Therefore when $t = 0$ we only need consider $c \neq 0$, and all nonzero c have the same generic behaviour.

$p = 2$

When $p = 2$, there is one irreducible representation of S_2 so we fix $\tau = \texttt{triv}$ and calculate the irreducible module $L_{0,c}(\texttt{triv})$. We therefore seek singular vectors in the Verma module $M_{t,c}(\texttt{triv}) \cong \Bbbk[x]$.

Proposition 6.1.1. *When $p = 2$, the singular vector of smallest degree in $M_{0,c}(\texttt{triv})$ is x.*

Proof. The value of $D_{y_1-y_2}(x^k)$ at $t = 0$ is $D_{y_1-y_2}(x^k) = -c(1 - (-1)^k)x^{k-1}$. When $k = 1$ this expression reads $-2c$ which is zero in characteristic 2. By definition, singular vectors in $M_{0,c}(\texttt{triv})$ must have strictly positive degree therefore this is the smallest. □

Proposition 6.1.2. *When $p = 2$, the character of $L_{0,c}(\texttt{triv})$ is*

$$\chi_{L_{0,c}(\texttt{triv})}(z) = [\texttt{triv}].$$

Proof. The quotient of $M_{0,c}(\texttt{triv}) \cong \Bbbk[x]$ by (x), the submodule generated by the singular vector x, is 1-dimensional with basis $\{1\}$. The group S_2 acts trivially on constants, therefore the character of the irreducible module is $[\texttt{triv}]$. □

Corollary 6.1.3. *When $p = 2$, the Hilbert polynomial of $L_{0,c}(\texttt{triv})$ is 1.*

Proof. This follows from the previous proposition, as the dimension of $[\texttt{triv}]$ is 1. □

$p \geq 3$

When $p \geq 3$ there are two irreducible representations of S_2 which are $\tau = \texttt{triv}$ and $\tau = \texttt{sign}$. However, by Corollary 2.8.3 the character of $L_{t,c}(\texttt{sign})$ can be derived from the character of $L_{t,-c}(\texttt{triv})$ through multiplication by $[\texttt{sign}]$.

Proposition 6.1.4. *When $p \geq 3$ and $c \neq 0$, the singular vector of smallest degree in $M_{0,c}(\texttt{triv})$ is x^2.*

Proof. We have $D_{y_1-y_2}(x^k) = -c\big(1 - (-1)^k\big)x^{k-1}$. When $k = 1$ this expression reads $-2c$ which is nonzero in characteristic $p \geq 3$, therefore x is not a singular vector. However, when

$k = 2$ this expression reads $-c(1-(-1)^2)x = 0$, therefore x^2 is the singular vector of smallest degree in $M_{0,c}(\texttt{triv})$. □

Proposition 6.1.5. *When $p \geq 3$ and $c \neq 0$, the character of $L_{0,c}(\texttt{triv})$ is*

$$\chi_{L_{0,c}(\texttt{triv})}(z) = [\texttt{triv}] + [\texttt{sign}]z.$$

Proof. The quotient of $M_{0,c}(\texttt{triv}) \cong \Bbbk[x]$ by (x^2), the submodule generated by the singular vector x^2, is 2-dimensional with basis $\{1, x\}$. By Proposition 6.1.4, x is not singular, therefore this quotient is irreducible because it does not contain and singular vectors hence has no proper submodules. The group S_2 acts trivially on constants, and x spans a $\texttt{sign}$ representation of S_2 in degree 1, therefore the character of this irreducible quotient is $[\texttt{triv}] + [\texttt{sign}]z$. □

Corollary 6.1.6. *When $p \geq 3$ and $c \neq 0$, the Hilbert polynomial of $L_{0,c}(\texttt{triv})$ is*

$$Hilb_{L_{0,c}(\texttt{triv})}(z) = 1 + z.$$

Proof. This follows from the previous proposition, as $\texttt{triv}$ and $\texttt{sign}$ are both 1-dimensional. □

Proposition 6.1.7. *When $p \geq 3$ and $c \neq 0$, the character of $L_{0,c}(\texttt{sign})$ is*

$$\chi_{L_{0,c}(\texttt{sign})}(z) = [\texttt{sign}] + [\texttt{triv}]z.$$

Proof. We have $\chi_{L_{0,c}(\texttt{triv})} = \chi_{L_{0,-c}(\texttt{triv})}$ because all nonzero c behave generically equivalent. Now

$$\begin{aligned}
\chi_{L_{0,c}(\texttt{sign})}(z) &= \chi_{L_{0,-c}(\texttt{triv})}(z) \cdot [\texttt{sign}] \\
&= \big([\texttt{triv}] + [\texttt{sign}]z\big) \cdot [\texttt{sign}] \\
&= [\texttt{triv} \otimes \texttt{sign}] + [\texttt{sign} \otimes \texttt{sign}]z \\
&= [\texttt{sign}] + [\texttt{triv}]z.
\end{aligned}$$

□

Corollary 6.1.8. *When $p \geq 3$ and $c \neq 0$, the Hilbert polynomial of $L_{0,c}(\texttt{sign})$ is*

$$Hilb_{L_{0,c}(\texttt{sign})}(z) = 1 + z.$$

6.2 $t = 1$

$p = 2$

We fix $\tau = \mathtt{triv}$ as this is the only irreducible representation of S_2 when $p = 2$.

Proposition 6.2.1. *When $p = 2$, the singular vector of smallest degree in $M_{1,c}(\mathtt{triv})$ is x^2, for all $c \in \mathbb{k}$.*

Proof. We have $D_{y_1-y_2}(x^k) = \big(k - c(1-(-1)^k)\big)x^{k-1}$. When $k = 1$ this reads $1 - 2c$ which is nonzero in characteristic 2, therefore x is not a singular vector. When $k = 2$ this reads $\big(2-c(1-(-1)^2)\big)x = 2x = 0$ in characteristic 2, therefore x^2 is the singular vector of smallest degree in $M_{1,c}(S_2, \mathfrak{h}, \mathtt{triv})$. ☐

Corollary 6.2.2. *When $p = 2$, all c are generic in $M_{1,c}(\mathtt{triv})$.*

Proof. The singular vector x^2 does not depend on c. The only possible values of c which could be special are $c = 0$ and $c = 1$ but neither of these values make x a singular vector. ☐

Proposition 6.2.3. *When $p = 2$, the character of $L_{1,c}(\mathtt{triv})$ is*

$$\chi_{L_{1,c}(S_2,\mathfrak{h},\mathtt{triv})}(z) = [\mathtt{triv}] + [\mathtt{triv}]z$$

for all $c \in \mathbb{k}$.

Proof. The quotient of $M_{1,c}(\mathtt{triv}) \cong \mathbb{k}[x]$ by (x^2) is 2-dimensional with basis $\{1, x\}$. The group S_2 acts trivially on constants, and x spans a $\mathtt{triv}$ representation of S_2 in degree 1, therefore the character of the irreducible module is $[\mathtt{triv}] + [\mathtt{triv}]z$. ☐

Corollary 6.2.4. *When $p = 2$, the Hilbert polynomial of $L_{1,c}(\mathtt{triv})$ is*

$$Hilb_{L_{1,c}(\mathtt{triv})}(z) = 1 + z$$

for all $c \in \mathbb{k}$.

$p \geq 3$

When $c = 0$ the singular vector of $M_{1,0}(\tau)$ is $x^p \otimes v$ for any $v \in \tau$ as explained in Proposition 2.6.13 and this matches the result stated in Theorem 6.0.3. We therefore suppose $c \neq 0$ and determine singular vectors in $M_{1,c}(\tau)$ when $p \geq 3$. We first consider $\tau = \mathtt{triv}$ and can afterwards derive the solution for $\mathtt{sign}$. We know that x^{2p} is singular for all c because $\sigma_2 = x^2$ is an S_2-invariant so its p^{th} power is singular. We can also see that when $k = 2p$ the Dunkl operator in (6.0.2) is equal to zero and does not depend on c. We are therefore interested in the question of existence of any singular vectors of degree strictly less than $2p$.

Proposition 6.2.5. *When $p \geq 3$, if $k < 2p$ and x^k is singular in $M_{1,c}(\mathtt{triv})$ then k is odd.*

Proof. Let x^k be singular with $0 < k < 2p$, so $D_{y_1-y_2}(x^k) = \big(k - c(1-(-1)^k)\big)x^{k-1} = 0$. Suppose k is even, then this expression reads $kx^{k-1} = 0$ which only holds if $p \mid k$. As $0 < k < 2p$ we must have $k = p$ which is a contradiction, therefore k is odd. □

Proposition 6.2.6. *For $c \in \Bbbk$, if x^k is singular in $M_{1,c}(\mathtt{triv})$ with degree $k < 2p$, then $c \in \{0, 1, \ldots, p-1\}$ and*

$$k = \begin{cases} 2c + p & \text{if } 0 \leq c < p/2, \\ 2c - p & \text{if } p/2 < c < p. \end{cases}$$

Proof. By the previous proposition, if x^k is singular with $k < 2p$ then k is odd. For odd k, the Dunkl operator reads $D_{y_1-y_2}(x^k) = (k-2c)x^{k-1}$. Therefore when k is odd, x^k is singular if and only if $k - 2c = 0$ when considered as an equation over $\Bbbk$. This equation can only be satisfied for values of c where $c = 2^{-1}k$ and since k is an integer, we conclude that $c \in \{0, 1, \ldots, p-1\}$. When $k - 2c = 0$ we may write this as an equation in the integers, $k = 2c + mp$ for some integer m. We want $k < 2p$ to be the smallest positive odd integer which satisfies this condition. If $0 \leq c < p/2$ then $2c < p$ and the smallest odd positive integer solution to $k = 2c + mp$ is $k = 2c + p$. However, if $p/2 < c < p$ then $2c > p$ and $k = 2c - p$ is the smallest odd positive integer solution. □

The preceeding proposition shows that the only special values of $c \in \Bbbk$ for $M_{1,c}(\mathtt{triv})$ are those $c \in \mathbb{F}_p$ and this agrees with Proposition 5.2.1 ([Li14], Theorem 2.8), which says that any values of $c \notin \mathbb{F}_p$ are generic.

Proposition 6.2.7. *The character of $L_{1,c}(\mathtt{triv})$ for generic c is*

$$\chi_{L_{1,c}(\mathtt{triv})}(z) = [\mathtt{triv}] + [\mathtt{sign}]z + [\mathtt{triv}]z^2 + \cdots + [\mathtt{sign}]z^{2p-1}.$$

Proof. The previous propositions show that the only time we have singular vectors in degree less than $2p$ is for particular values of c. However for all other c, x^{2p} is generically the singular vector of smallest degree, therefore the irreducible module $L_{1,c}(\mathtt{triv})$ is the baby Verma module $N_{1,c}(\mathtt{triv})$ for generic c. The character of this module alternates between `triv` in even degree and `sign` in odd degree, and stops in degree $2p-1$ because x^{2p} is a singular vector for all c. □

Corollary 6.2.8. *The Hilbert polynomial of $L_{1,c}(\mathtt{triv})$ for generic c is*

$$Hilb_{L_{1,c}(\mathtt{triv})}(z) = 1 + z + z^2 + \cdots + z^{2p-1}.$$

Proposition 6.2.9. *The character of $L_{1,c}(\mathtt{triv})$ for $c \in \mathbb{F}_p$ is*

$$\chi_{L_{1,c}(\mathtt{triv})}(z) = [\mathtt{triv}] + [\mathtt{sign}]z + [\mathtt{triv}]z^2 + \cdots + [\mathtt{sign}]z^{k-1}$$

where

$$k = \begin{cases} 2c+p & \text{if } 0 \leq c < p/2, \\ 2c-p & \text{if } p/2 < c < p. \end{cases}$$

Proof. The Verma module $M_{1,c}(\texttt{triv})$ alternates between `triv` in even degree and `sign` in odd degree. The irreducible quotient stops in degree $k-1$ where k is the degree of the singular vector described in Proposition 6.2.6. □

Corollary 6.2.10. *The Hilbert polynomial of $L_{1,c}(\texttt{triv})$ for $c \in \mathbb{F}_p$ is*

$$Hilb_{L_{1,c}(\texttt{triv})}(z) = 1 + z + z^2 + \cdots + z^{k-1}$$

where

$$k = \begin{cases} 2c+p & \text{if } 0 \leq c < p/2, \\ 2c-p & \text{if } p/2 < c < p. \end{cases}$$

We can obtain results for $\tau = \texttt{sign}$ by considering Corollary 2.8.3 with $\chi_{L_{1,c}(\texttt{sign})}(z) = \chi_{L_{1,-c}(\texttt{triv})}(z) \cdot [\texttt{sign}]$.

Proposition 6.2.11. *The character of $L_{1,c}(\texttt{sign})$ for generic c is*

$$\chi_{L_{1,c}(\texttt{sign})}(z) = [\texttt{sign}] + [\texttt{triv}]z + [\texttt{sign}]z^2 + \cdots + [\texttt{triv}]z^{2p-1}.$$

Corollary 6.2.12. *The Hilbert polynomial of $L_{1,c}(\texttt{sign})$ for generic c is*

$$Hilb_{L_{1,c}(\texttt{sign})}(z) = 1 + z + z^2 + \cdots + z^{2p-1}.$$

Proposition 6.2.13. *The character of $L_{1,c}(\texttt{sign})$ for $c \in \mathbb{F}_p$ is*

$$\chi_{L_{1,c}(\texttt{triv})}(z) = [\texttt{sign}] + [\texttt{triv}]z + [\texttt{sign}]z^2 + \cdots + [\texttt{sign}]z^{k-1}$$

where

$$k = \begin{cases} -2c+p & \text{if } 0 \leq c < p/2, \\ -2c+3p & \text{if } p/2 < c < p. \end{cases}$$

Proof. The character of the Verma module $M_{1,c}(\texttt{sign})$ alternates between `sign` in even degree and `triv` in odd degree. The singular vector of $M_{1,c}(\texttt{sign})$ has the same degree as the singular vector of $M_{1,-c}(\texttt{triv})$. Let $c \in \{0, 1, \ldots, p-1\}$. If $c = 0$ then $-c = 0 \in \{0, 1, \ldots, p-1\}$ and by Proposition 6.2.9 the character is as stated since $2c + p = -2c + p$. If $c \neq 0$ then $-c = p - c \in \{1, \ldots, p-1\}$, and for $p - c$ in the range $0 < p - c < p/2$ rearranging gives $p/2 < c < p$, while for $p - c$ in the range $p/2 < p - c < p$ rearranging gives $0 < c < p/2$.

Substituting $p - c$ into the appropriate formula from Proposition 6.2.9 we get

$$k = \begin{cases} 2(p-c)+p & \text{if } 0 < p-c < p/2, \\ 2(p-c)-p & \text{if } p/2 < p-c < p, \end{cases}$$

which simplifies to

$$k = \begin{cases} -2c+p & \text{if } 0 < c < p/2, \\ -2c+3p & \text{if } p/2 < c < p. \end{cases}$$

□

Corollary 6.2.14. *The Hilbert polynomial of $L_{1,c}(\texttt{sign})$ for $c \in \mathbb{F}_p$ is*

$$Hilb_{L_{1,c}(\texttt{triv})}(z) = 1 + z + z^2 + \cdots + z^{k-1}$$

where

$$k = \begin{cases} -2c+p & \textit{if } 0 < c < p/2, \\ -2c+3p & \textit{if } p/2 < c \leq p. \end{cases}$$

Chapter 7

Bases for the Verma Modules $M_{t,c}(S_3, \mathfrak{h}, \tau)$

For the rest of the thesis, we fix $n = 3$. In order to do explicit computations with Verma modules, we will need to fix a basis for $S(\mathfrak{h}^*)$.

7.1 Choices of basis in $\mathfrak{h}^*$

In any characteristic we can use $\{\overline{x_1}, \overline{x_2}\}$ as a basis for $\mathfrak{h}^*$. We also have an element $\overline{x_3} \in \mathfrak{h}^*$ which satisfies $\overline{x_1} + \overline{x_2} + \overline{x_3} = 0$.

In characteristic $p \neq 3$, the standard representation `stand` is irreducible and isomorphic to $\mathfrak{h}^*$, and we also use $\{\overline{x_1}, \overline{x_2}\}$ as its basis.

In characteristic $p > 3$, in order to exploit some additional symmetries and reduce the number of computations we need to do, it is convenient to use a basis which is well-behaved when restricting from S_3 to $S_2 = \{e, (12)\}$. As explained in Section 3.2, when $p \nmid n$ we can realise $\mathfrak{h}^*$ as a subrepresentation of V^* consisting of all $a_1x_1 + a_2x_2 + a_3x_3 \in V^*$ with $a_1 + a_2 + a_3 = 0$. Now we define the rescaled Young basis of $\mathfrak{h}^*$ by

$$b_+ = x_1 + x_2 - 2x_3,$$
$$b_- = x_1 - x_2.$$

Let s_i denote the simple transposition $(i, i+1)$. The basis $\{b_+, b_-\}$ of $\mathfrak{h}^*$ satisfies:

$$s_1.b_+ = b_+ \qquad s_1.b_- = -b_- \qquad b_- = \frac{2}{3}\left(s_2 + \frac{1}{2}e\right)b_+,$$

which shows it is a rescaling of the usual Young basis (by a factor of 1/2 and 1/3).

7.2 Verma modules, their bases and characters

We will need the following combinatorial lemma.

Lemma 7.2.1. *For any $k \in \mathbb{N}_0$ the number of non-negative integral solutions $(a, b) \in \mathbb{N}_0^2$ to the equation $2a + 3b = k$ equals*

$$\begin{cases} \lfloor \frac{k}{6} \rfloor + 1 & k \text{ even}, \\ \lfloor \frac{k-3}{6} \rfloor + 1 & k \text{ odd}. \end{cases}$$

Proof. We will parametrise the solutions for later use. First assume $k = 2k'$ is even for some $k' \in \mathbb{N}_0$. The equation $2a + 3b = k$ then becomes

$$2a + 3b = 2k'$$

so b is even. Write $b = 2j$ for some $j \geq 0$. The equation then becomes

$$a = k' - 3j$$

which gives another condition $j \leq \frac{k'}{3}$. So, the set of solutions $\{(a, b)\}$ is parametrised as

$$\left\{ \left(\frac{k}{2} - 3j, 2j \right) \mid 0 \leq j \leq \frac{k}{6} \right\}$$

and their total number is $\lfloor \frac{k}{6} \rfloor + 1$.

Now assume $k = 2k' + 1$ is odd. The equation $2a + 3b = k$ then becomes

$$2a + 3b = 2k' + 1$$

so b is odd. Write $b = 2j + 1$ for some $j \geq 0$. The equation then becomes

$$a = k' - 1 - 3j$$

which gives another condition $j \leq \frac{k'-1}{3}$. So, the set of solutions $\{(a, b)\}$ is parametrised as

$$\left\{ \left(\frac{k}{2} - 1 - 3j, 2j + 1 \right) \mid 0 \leq j \leq \frac{k-3}{6} \right\}$$

and their total number is $\lfloor \frac{k-3}{6} \rfloor + 1$. □

$p > 3$

Recall that $\sigma_2, \ldots, \sigma_n$ are symmetric polynomials which generate the algebra of invariants $S(\mathfrak{h}^*)^{S_n}$ in characteristic $p \nmid n$, as outlined in Proposition 3.3.12.

Theorem 7.2.2. *In characteristic $p > 3$, $M^k_{t,c}(\mathtt{triv}) \cong S^k(\mathfrak{h}^*)$ is a direct sum of the following irreducible S_3 representations:*

- *for every $a, b \in \mathbb{N}_0$ satisfying $2a + 3b = k$, a subrepresentation isomorphic to* `triv` *with a basis*
$$\{\sigma_2^a \sigma_3^b\};$$
- *for every $a, b \in \mathbb{N}_0$ satisfying $2a + 3b = k - 1$, a subrepresentation isomorphic to* `stand` *with a basis*
$$\{\sigma_2^a \sigma_3^b b_+,\ \sigma_2^a \sigma_3^b b_-\};$$
- *for every $a, b \in \mathbb{N}_0$ satisfying $2a + 3b = k - 2$, a subrepresentation isomorphic to* `stand` *with a basis*
$$\{\sigma_2^a \sigma_3^b(-b_+^2 + 3b_-^2),\ \sigma_2^a \sigma_3^b(2b_+b_-)\}.$$
- *for every $a, b \in \mathbb{N}_0$ satisfying $2a + 3b = k - 3$, a subrepresentation isomorphic to* `sign` *with a basis*
$$\{q\sigma_2^a \sigma_3^b\}$$
where $q = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ is the Vandermonde polynomial.

Proof. The S_3 representation $S^k(\mathfrak{h}^*)$ is a direct sum of its isotypic components, so it is enough to show that the above vectors are in the correct isotypic components and are a basis.

The space of invariants in $S(\mathfrak{h}^*)$ is the polynomial algebra in σ_2 and σ_3 (see Proposition 3.3.12). Closely related to the symmetric polynomials are the antisymmetric polynomials, $f \in S(\mathfrak{h}^*)$ which transform as the `sign` representation. By considering each transposition in turn and writing out $(ij).f = -f$, one sees that f is divisible by $x_i - x_j$ for all $i \neq j$, and consequently that f is divisible by the Vandermonde polynomial $q = \prod_{i<j}(x_i - x_j)$. It then quickly follows that any antisymmetric polynomial f is of the form $f = q \cdot f'$, where f' is a symmetric polynomial.

One can check directly that for every choice of a, b the span of $\{\sigma_2^a \sigma_3^b b_+, \sigma_2^a \sigma_3^b b_-\}$ is an S_3 subrepresentation isomorphic to `stand` with the isomorphism given by $b_\pm \mapsto \sigma_2^a \sigma_3^b b_\pm$, and that for every choice of a, b the span of $\{\sigma_2^a \sigma_3^b(-b_+^2 + 3b_-^2),\ \sigma_2^a \sigma_3^b(2b_+b_-)\}$ is an S_3 subrepresentation isomorphic to `stand` with the isomorphism given by $b_+ \mapsto \sigma_2^a \sigma_3^b(-b_+^2 + 3b_-^2)$, $b_- \mapsto \sigma_2^a \sigma_3^b(2b_+b_-)$.

Next, let us show that the set

$$\{\sigma_2^a \sigma_3^b b_+,\ \sigma_2^a \sigma_3^b b_- \mid 2a + 3b = k - 1\} \cup \{\sigma_2^a \sigma_3^b(-b_+^2 + 3b_-^2),\ \sigma_2^a \sigma_3^b(2b_+b_-) \mid 2a + 3b = k - 2\}$$

is linearly independent. Assume a nontrivial linear combination of these vectors is zero. Gathering the terms, this can be written as

$$f_1 \cdot b_+ + f_2 \cdot b_- + f_3 \cdot (-b_+^2 + 3b_-^2) + f_4 \cdot 2b_+b_- = 0,$$

where f_1, f_2 are symmetric polynomials of degree $k-1$, f_3, f_4 are symmetric polynomials of degree $k-2$, and f_1, f_2, f_3, f_4 are not all zero. Recall that b_+ and $(-b_+^2 + 3b_-^2)$ are fixed by s_1, while the action of s_1 on b_- and $2b_+b_-$ is multiplication by -1. If f_2, f_4 are not both zero, let us apply the projection $\frac{1}{2}(\mathrm{id} - s_1)$ to get

$$f_2 \cdot b_- + f_4 \cdot 2b_+b_- = 0.$$

If f_2, f_4 are both zero, the expression is

$$f_1 \cdot b_+ + f_3 \cdot (-b_+^2 + 3b_-^2) = 0,$$

which after applying $\frac{2}{3}\left(s_2 + \frac{1}{2}\mathrm{id}\right)$ becomes

$$f_1 \cdot b_- + f_3 \cdot 2b_+b_- = 0.$$

So, in either case we can assume we have a linear combination

$$f_2 \cdot b_- + f_4 \cdot 2b_+b_- = 0$$

with f_2, f_4 symmetric and not both zero. Dividing by b_- we get

$$f_2 = -f_4 \cdot 2b_+,$$

which is an equality between a symmetric polynomial and a polynomial which is not symmetric, so it only holds if f_2, f_4 are both zero. This is a contradiction, and we conclude that the set

$$\{\sigma_2^a\sigma_3^b b_+,\ \sigma_2^a\sigma_3^b b_- \mid 2a+3b = k-1\} \cup \{\sigma_2^a\sigma_3^b(-b_+^2 + 3b_-^2),\ \sigma_2^a\sigma_3^b(2b_+b_-) \mid 2a+3b = k-2\}$$

is linearly independent.

We have now seen that the vectors from the statement of the theorem lie in the correct isotypic components and are linearly independent. Their number is equal to

$$\begin{aligned}\text{number of vectors } &= \left|\{(a,b) \in \mathbb{N}_0^2 \mid 2a+3b = k\}\right| + 2\left|\{(a,b) \in \mathbb{N}_0^2 \mid 2a+3b = k-1\}\right| + \\ &\quad + 2\left|\{(a,b) \in \mathbb{N}_0^2 \mid 2a+3b = k-2\}\right| + \left|\{(a,b) \in \mathbb{N}_0^2 \mid 2a+3b = k-3\}\right|.\end{aligned}$$

By Lemma 7.2.1, if k is even, this is equal to

$$\begin{aligned}\text{number of vectors } &= \left\lfloor \frac{k}{6} \right\rfloor + 1 + 2\left(\left\lfloor \frac{k-4}{6} \right\rfloor + 1\right) + 2\left(\left\lfloor \frac{k-2}{6} \right\rfloor + 1\right) + \left\lfloor \frac{k-6}{6} \right\rfloor + 1 \\ &= 5 + 2\left(\left\lfloor \frac{k}{6} \right\rfloor + \left\lfloor \frac{k-2}{6} \right\rfloor + \left\lfloor \frac{k-4}{6} \right\rfloor\right) \\ &= 5 + 2\left(\frac{k}{2} - 2\right) \\ &= k + 1 = \dim S^k(\mathfrak{h}^*).\end{aligned}$$

If k is odd, then by Lemma 7.2.1

$$\begin{aligned}\text{number of vectors } &= \left\lfloor \frac{k-3}{6} \right\rfloor + 1 + 2\left(\left\lfloor \frac{k-1}{6} \right\rfloor + 1\right) + 2\left(\left\lfloor \frac{k-5}{6} \right\rfloor + 1\right) + \left\lfloor \frac{k-3}{6} \right\rfloor + 1 \\ &= 6 + 2\left(\left\lfloor \frac{k-1}{6} \right\rfloor + \left\lfloor \frac{k-3}{6} \right\rfloor + \left\lfloor \frac{k-5}{6} \right\rfloor\right) \\ &= 6 + 2\left(\frac{k-1}{2} - 2\right) \\ &= k + 1 = \dim S^k(\mathfrak{h}^*).\end{aligned}$$

So, the vectors in the statement of the theorem are a linearly independent set of size equal to the dimension of the vector space, so they are a basis. □

We wish to find nice bases of Verma modules to work with, which are compatible with their decomposition as S_3 representations, and in which we can reasonably compute Dunkl operators. For $\tau = \mathtt{triv}$, we have $M_{t,c}(\mathtt{triv}) \cong S(\mathfrak{h}^*) \otimes \mathtt{triv} \cong S(\mathfrak{h}^*)$ as an S_3 representation, so Theorem 7.2.2 gives us such a basis. A similar computation works when $\tau = \mathtt{sign}$. For $\tau = \mathtt{stand}$, we have $M_{t,c}(\mathtt{stand}) \cong S(\mathfrak{h}^*) \otimes \mathtt{stand}$ as an S_3 representation, so we need to understand how the above basis of $S(\mathfrak{h}^*)$ behaves after taking a tensor product with $\mathtt{stand}$. The following lemma is a standard exercise in representations of finite groups.

Lemma 7.2.3. *Let $p > 3$. As S_3 representations,*

1. $\mathtt{triv} \otimes \mathtt{stand} \cong \mathtt{stand}$ *tautologically;*
2. $\mathtt{sign} \otimes \mathtt{stand} \cong \mathtt{stand}$ *with the isomorphism given by*

$$1_{\mathtt{sign}} \otimes b_+ \mapsto -3b_-, \quad 1_{\mathtt{sign}} \otimes b_- \mapsto b_+;$$

3. $\mathtt{stand} \otimes \mathtt{stand} \cong \mathtt{triv} \oplus \mathtt{sign} \oplus \mathtt{stand}$*; with a compatible basis given by*
   - $b_+ \otimes b_+ + 3b_- \otimes b_-$, *spanning a subrepresentation isomorphic to* $\mathtt{triv}$*;*
   - $-b_- \otimes b_+ + b_+ \otimes b_-$, *spanning a subrepresentation isomorphic to* $\mathtt{sign}$*;*

- $-b_+ \otimes b_+ + 3b_- \otimes b_-$, $b_- \otimes b_+ + b_+ \otimes b_-$ *spanning a subrepresentation isomorphic to* `stand`*, with the isomorphism given by*

$$b_+ \mapsto -b_+ \otimes b_+ + 3b_- \otimes b_-, \quad b_- \mapsto b_+ \otimes b_- + b_- \otimes b_+.$$

Proof. (1) is tautologically true and requires no proof.
For (2), a map of representations $\mathtt{sign} \otimes \mathtt{stand} \to \mathtt{stand}$ sends each element of the basis $\{1_{\mathtt{sign}} \otimes b_+,\ 1_{\mathtt{sign}} \otimes b_-\}$ of the domain to some linear combination of the basis $\{b_+,\ b_-\}$ of the codomain, and commutes with the action of S_3. Considering the restricted action of S_2 it is clear that $1_{\mathtt{sign}} \otimes b_+ \mapsto Ab_-$ and $1_{\mathtt{sign}} \otimes b_- \mapsto Bb_+$ for some constants $A, B \in \Bbbk$. Now

$$\begin{aligned}(23).(1_{\mathtt{sign}} \otimes b_-) &= \big((23).1_{\mathtt{sign}}\big) \otimes \big((23).b_-\big)\\ &= -1_{\mathtt{sign}} \otimes \left(\frac{b_+ + b_-}{2}\right)\\ &= \frac{-1}{2}\big(1_{\mathtt{sign}} \otimes b_+ + 1_{\mathtt{sign}} \otimes b_-\big).\end{aligned}$$

This implies

$$(23).Bb_+ = B\left(\frac{-b_+ + 3b_-}{2}\right) = \frac{-1}{2}\big(Ab_- + Bb_+\big)$$

hence $A = -3B$ as claimed.
As for (3), given a finite group G, a vector space W over an algebraically closed field $\Bbbk$ of characteristic $p \nmid |G|$, a representation $\rho : G \to \mathrm{End}(W)$, and an irreducible representation $\tau \in \widehat{G}$, the projection to the τ isotypic component of ρ is given by the map

$$\pi_\tau : W \to W|_\tau, \quad \pi_\tau = \frac{\dim \tau}{|G|} \sum_{g \in G} \chi_\tau(g)\rho(g).$$

Therefore the projection to the `triv` isotypic component of $\mathtt{stand} \otimes \mathtt{stand}$ is given by the map

$$\pi_{\mathtt{triv}} = \frac{1}{6}\big(e + (12) + (13) + (23) + (123) + (132)\big),$$

the projection to the `sign` isotypic component of $\mathtt{stand} \otimes \mathtt{stand}$ is given by the map

$$\pi_{\mathtt{sign}} = \frac{1}{6}\big(e - (12) - (13) - (23) + (123) + (132)\big),$$

and the projection to the `stand` isotypic component of $\mathtt{stand} \otimes \mathtt{stand}$ is given by the map

$$\pi_{\mathtt{stand}} = \frac{1}{3}\big(2 \cdot e - (123) - (132)\big).$$

The matrices for the action of S_3 on $\texttt{stand} \otimes \texttt{stand}$ are given by

$$[e] = I_4$$

$$[(12)] = \begin{array}{c} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array} \quad \begin{array}{l} \text{cols: } b_+\otimes b_+,\ b_-\otimes b_+,\ b_+\otimes b_-,\ b_-\otimes b_- \\ \text{rows: } b_+\otimes b_+,\ b_-\otimes b_+,\ b_+\otimes b_-,\ b_-\otimes b_- \end{array}$$

$$[(13)] = \frac{1}{4}\left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 3 & -1 & 3 & -1 \\ 3 & 3 & -1 & -1 \\ 9 & -3 & -3 & 1 \end{array}\right] \quad \begin{array}{l} \text{cols: } b_+\otimes b_+,\ b_-\otimes b_+,\ b_+\otimes b_-,\ b_-\otimes b_- \\ \text{rows: } b_+\otimes b_+,\ b_-\otimes b_+,\ b_+\otimes b_-,\ b_-\otimes b_- \end{array}$$

$$[(23)] = \frac{1}{4}\left[\begin{array}{cccc} 1 & -1 & -1 & 1 \\ -3 & -1 & 3 & 1 \\ -3 & 3 & -1 & 1 \\ 9 & 3 & 3 & 1 \end{array}\right] \quad \begin{array}{l} \text{cols: } b_+\otimes b_+,\ b_-\otimes b_+,\ b_+\otimes b_-,\ b_-\otimes b_- \\ \text{rows: } b_+\otimes b_+,\ b_-\otimes b_+,\ b_+\otimes b_-,\ b_-\otimes b_- \end{array}$$

$$[(123)] = \frac{1}{4}\left[\begin{array}{cccc} 1 & -1 & -1 & 1 \\ 3 & 1 & -3 & -1 \\ 3 & -3 & 1 & -1 \\ 9 & 3 & 3 & 1 \end{array}\right] \quad \begin{array}{l} \text{cols: } b_+\otimes b_+,\ b_-\otimes b_+,\ b_+\otimes b_-,\ b_-\otimes b_- \\ \text{rows: } b_+\otimes b_+,\ b_-\otimes b_+,\ b_+\otimes b_-,\ b_-\otimes b_- \end{array}$$

$$[(132)] = \frac{1}{4}\left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ -3 & 1 & -3 & 1 \\ -3 & -3 & 1 & 1 \\ 9 & -3 & -3 & 1 \end{array}\right] \quad \begin{array}{l} \text{cols: } b_+\otimes b_+,\ b_-\otimes b_+,\ b_+\otimes b_-,\ b_-\otimes b_- \\ \text{rows: } b_+\otimes b_+,\ b_-\otimes b_+,\ b_+\otimes b_-,\ b_-\otimes b_- \end{array}$$

Therefore the projection $\pi_{\texttt{triv}}$ can be written as the matrix

$$[\pi_{\texttt{triv}}] = \frac{1}{6}\begin{array}{c} \begin{array}{cccc} \scriptstyle b_+\otimes b_+ & \scriptstyle b_-\otimes b_+ & \scriptstyle b_+\otimes b_- & \scriptstyle b_-\otimes b_- \end{array} \\ \left[\begin{array}{cccc} 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 3 \end{array}\right] \end{array} \begin{array}{l} \scriptstyle b_+\otimes b_+ \\ \scriptstyle b_-\otimes b_+ \\ \scriptstyle b_+\otimes b_- \\ \scriptstyle b_-\otimes b_- \end{array}$$

hence the $\texttt{triv}$ isotypic component of $\texttt{stand} \otimes \texttt{stand}$ is spanned by $b_+ \otimes b_+ + 3b_- \otimes b_-$. Similarly, the projection $\pi_{\texttt{sign}}$ can be written as the matrix

$$[\pi_{\texttt{sign}}] = \frac{1}{6}\begin{array}{c} \begin{array}{cccc} \scriptstyle b_+\otimes b_+ & \scriptstyle b_-\otimes b_+ & \scriptstyle b_+\otimes b_- & \scriptstyle b_-\otimes b_- \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \scriptstyle b_+\otimes b_+ \\ \scriptstyle b_-\otimes b_+ \\ \scriptstyle b_+\otimes b_- \\ \scriptstyle b_-\otimes b_- \end{array}$$

hence the $\texttt{sign}$ isotypic component of $\texttt{stand} \otimes \texttt{stand}$ is spanned by $-b_- \otimes b_+ + b_+ \otimes b_-$. Finally, the projection $\pi_{\texttt{stand}}$ can be written as the matrix

$$[\pi_{\texttt{stand}}] = \frac{1}{2}\begin{array}{c} \begin{array}{cccc} \scriptstyle b_+\otimes b_+ & \scriptstyle b_-\otimes b_+ & \scriptstyle b_+\otimes b_- & \scriptstyle b_-\otimes b_- \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \scriptstyle b_+\otimes b_+ \\ \scriptstyle b_-\otimes b_+ \\ \scriptstyle b_+\otimes b_- \\ \scriptstyle b_-\otimes b_- \end{array}$$

hence the $\texttt{stand}$ isotypic component of $\texttt{stand} \otimes \texttt{stand}$ is spanned by $-b_+ \otimes b_+ + 3b_- \otimes b_-$ and $b_- \otimes b_+ + b_+ \otimes b_-$. □

Now by putting together Theorem 7.2.2 and Lemma 7.2.3 we immediately get the following theorem.

Theorem 7.2.4. *In characteristic $p > 3$, $M^k_{t,c}(\texttt{stand}) \cong S^k(\mathfrak{h}^*) \otimes \texttt{stand}$ is a direct sum of the following irreducible S_3 representations:*

- *for every $a, b \in \mathbb{N}_0$ satisfying $2a+3b = k$, a subrepresentation isomorphic to* $\texttt{stand}$ *with a basis*
$$\{\sigma_2^a\sigma_3^b \otimes b_+,\ \sigma_2^a\sigma_3^b \otimes b_-\};$$
- *for every $a, b \in \mathbb{N}_0$ satisfying $2a + 3b = k - 3$, a subrepresentation isomorphic to* $\texttt{stand}$

with a basis

$$\{q\sigma_2^a\sigma_3^b \otimes b_-, \ \frac{-1}{3} \cdot q\sigma_2^a\sigma_3^b \otimes b_+\};$$

- *for every* $a, b \in \mathbb{N}_0$ *satisfying* $2a + 3b = k - 1$*, a subrepresentation isomorphic to* `triv` *with a basis*

$$\{\sigma_2^a\sigma_3^b \cdot (b_+ \otimes b_+ + 3b_- \otimes b_-)\};$$

- *for every* $a, b \in \mathbb{N}_0$ *satisfying* $2a + 3b = k - 1$*, a subrepresentation isomorphic to* `sign` *with a basis*

$$\{\sigma_2^a\sigma_3^b \cdot (-b_- \otimes b_+ + b_+ \otimes b_-)\};$$

- *for every* $a, b \in \mathbb{N}_0$ *satisfying* $2a + 3b = k - 1$*, a subrepresentation isomorphic to* `stand` *with a basis*

$$\{\sigma_2^a\sigma_3^b \cdot (-b_+ \otimes b_+ + 3b_- \otimes b_-), \ \sigma_2^a\sigma_3^b \cdot (b_- \otimes b_+ + b_+ \otimes b_-)\};$$

- *for every* $a, b \in \mathbb{N}_0$ *satisfying* $2a + 3b = k - 2$*, a subrepresentation isomorphic to* `triv` *with a basis*

$$\{\sigma_2^a\sigma_3^b \cdot \big((-b_+^2 + 3b_-^2) \otimes b_+ + 3 \cdot (2b_+b_-) \otimes b_-\big)\}.$$

- *for every* $a, b \in \mathbb{N}_0$ *satisfying* $2a + 3b = k - 2$*, a subrepresentation isomorphic to* `sign` *with a basis*

$$\{\sigma_2^a\sigma_3^b \cdot \big(-(2b_+b_-) \otimes b_+ + (-b_+^2 + 3b_-^2) \otimes b_-\big)\}.$$

- *for every* $a, b \in \mathbb{N}_0$ *satisfying* $2a + 3b = k - 2$*, a subrepresentation isomorphic to* `stand` *with a basis*

$$\{\sigma_2^a\sigma_3^b\cdot\big(-(-b_+^2 + 3b_-^2) \otimes b_+ + 3(2b_+b_-) \otimes b_-\big), \ \sigma_2^a\sigma_3^b\cdot\big((2b_+b_-) \otimes b_+ + (-b_+^2 + 3b_-^2) \otimes b_-\big)\}.$$

Wherever in this statement a basis of `stand` *is given as* $\{u, v\}$*, the map* $b_+ \mapsto u$*,* $b_- \mapsto v$ *is an* S_3 *isomorphism.*

Corollary 7.2.5. *Assume* $p > 3$*.*

1. *The character of the graded* S_3 *representation* $S(\mathfrak{h}^*)$ *is*

$$\chi_{S(\mathfrak{h}^*)}(z) = \frac{1}{(1 - z^2)(1 - z^3)} \left([\texttt{triv}] + [\texttt{stand}](z + z^2) + [\texttt{sign}]z^3\right).$$

2. *The characters of Verma modules for* $H_{t,c}(S_3, \mathfrak{h})$ *are given by*

$$\chi_{M_{t,c}(\texttt{triv})}(z) = \chi_{S(\mathfrak{h}^*)}(z)$$

$$\chi_{M_{t,c}(\texttt{sign})}(z) = \frac{1}{(1-z^2)(1-z^3)}\left([\texttt{sign}] + [\texttt{stand}](z+z^2) + [\texttt{triv}]z^3\right)$$
$$\chi_{M_{t,c}(\texttt{stand})}(z) = \frac{1}{(1-z^2)(1-z^3)}\left([\texttt{stand}](1+z+z^2+z^3) + ([\texttt{triv}]+[\texttt{sign}])(z+z^2)\right).$$

3. *The characters of baby Verma modules for $H_{t,c}(S_3, \mathfrak{h})$ are given by*

$$\chi_{N_{0,c}(\tau)}(z) = \chi_{M_{t,c}(\tau)}(z)(1-z^2)(1-z^3).$$
$$\chi_{N_{1,c}(\tau)}(z) = \chi_{M_{t,c}(\tau)}(z)(1-z^{2p})(1-z^{3p}).$$

Proof. Using the decomposition from Theorem 7.2.2 we get

$$\chi_{S(\mathfrak{h}^*)}(z) = \sum_{a,b} z^{2a+3b}[\texttt{triv}] + \sum_{a,b} z^{2a+3b+1}[\texttt{stand}] + \sum_{a,b} z^{2a+3b+2}[\texttt{stand}] + \sum_{a,b} z^{2a+3b+3}[\texttt{sign}]$$
$$= \frac{1}{(1-z^2)(1-z^3)}\left([\texttt{triv}] + [\texttt{stand}](z+z^2) + [\texttt{sign}]z^3\right).$$

The characters of Verma modules then follow from $M_{t,c}(\tau) = S(\mathfrak{h}^*) \otimes \tau$ and Lemma 7.2.3, and the characters of baby Verma modules follow from the fact that $N_{t,c}(\tau)$ is a quotient of $M_{t,c}(\tau)$ by $\sigma_i \otimes \tau$ (when $t=0$) or by $\sigma_i^p \otimes \tau$ (when $t=1$).

□

Recall from Definition 2.6.12 that $S^{(p)}(\mathfrak{h}^*)$ is the quotient of $S(\mathfrak{h}^*)$ by the $S(\mathfrak{h}^*)$-submodule generated by $(x_i - x_j)^p$ for all i, j (or equivalently by b_+^p, b_-^p). It is a finite-dimensional representation of dimension p^2.

Corollary 7.2.6. *For $p > 3$, the character of the graded S_3 representation $S^{(p)}(\mathfrak{h}^*)$ is*

$$\chi_{S^{(p)}(\mathfrak{h}^*)}(z) = \chi_{S(\mathfrak{h}^*)}(z) \cdot ([\texttt{triv}] - [\texttt{stand}]z^p + [\texttt{sign}]z^{2p})$$
$$= \frac{1}{(1-z^2)(1-z^3)}\left([\texttt{triv}] + [\texttt{stand}](z+z^2) + [\texttt{sign}]z^3\right)(1 - z^p[\texttt{stand}] + z^{2p}[\texttt{sign}]).$$

Proof. The character of $S^{(p)}(\mathfrak{h}^*)$ is the character of $S(\mathfrak{h}^*)$ minus the character of the $S(\mathfrak{h}^*)$-submodule generated by b_+^p, b_-^p.
The vectors b_+^p, b_-^p span a representation isomorphic to `stand` with the isomorphism

$$\varphi : \texttt{stand} \to \operatorname{span}\{b_+^p, b_-^p\}$$

given by

$$\varphi(b_\pm) = b_\pm^p.$$

By the universal mapping property of the induced module $M_{t,c}(\texttt{stand})$, the homomorphism ϕ of S_3 representations extends to a homomorphism, also denoted φ, of graded $H_{t,c}(S_3, \mathfrak{h})$

representations

$$\varphi : M_{t,c}(\texttt{stand})[-p] \to M_{t,c}(\texttt{stand})$$

defined on the generators as

$$\varphi(1 \otimes b_\pm) = b_\pm^p$$

where $M[k]$ denotes a grading shift by k per Definition 2.6.4. The kernel of this map is the set of all $S(\mathfrak{h}^*)$ multiples of $b_+^p \otimes b_-^p - b_-^p \otimes b_+^p \in M_{t,c}(\texttt{stand})[-p]$ and it is isomorphic to $M_{t,c}(\texttt{sign})[-2p]$. Hence

$$\begin{aligned}
\chi_{S^{(p)}(\mathfrak{h}^*)}(z) &= \chi_{S^{(p)}(\mathfrak{h}^*)}(z) - \chi_{\langle b_+^p, b_-^p\rangle}(z) \\
&= \chi_{S^{(p)}(\mathfrak{h}^*)}(z) - \chi_{\mathrm{im}(\varphi)}(z) \\
&= \chi_{S^{(p)}(\mathfrak{h}^*)}(z) - \chi_{M_{t,c}(\texttt{stand})[-p]}(z) + \chi_{\ker(\varphi)}(z) \\
&= \chi_{S^{(p)}(\mathfrak{h}^*)}(z) - \chi_{M_{t,c}(\texttt{stand})[-p]}(z) + \chi_{M_{t,c}(\texttt{sign})[-2p]}(z)
\end{aligned}$$

The result then follows from Corollary 7.2.5, noting that $\chi_{M[-k]}(z) = z^k \chi_M(z)$. □

$p = 2$

In characteristic 2 the rescaled Young basis $\{b_+, b_-\}$ does not make sense. Specifically, this basis relies on restricting the standard representation of S_3 to $S_2 = \{e, s_1\}$ and decomposing it into the trivial and sign representation of S_2. This works in characteristics $p = 0$ and $p > 3$, but in characteristic 2 the trivial and the sign representations are isomorphic, and the standard representation of S_3, when restricted to S_2, becomes an indecomposable extension of two copies of the trivial representation. Instead, in characteristic 2 we use the basis $\{\overline{x_1}, \overline{x_2}\}$ of the standard representation, with $\overline{x_3} = \overline{x_1} + \overline{x_2}$. The elementary symmetric polynomials in this case are

$$\begin{aligned}
\overline{\sigma_2} &= \overline{x_1}^2 + \overline{x_1 x_2} + \overline{x_2}^2 \\
\overline{\sigma_3} &= \overline{x_1}^2 \overline{x_2} + \overline{x_1 x_2}^2.
\end{aligned}$$

The analogue of Theorem 7.2.2 in characteristic 2 is the following theorem.

Theorem 7.2.7. *In characteristic $p = 2$, $S^k(\mathfrak{h}^*)$ is a direct sum of the following indecomposable S_3 representations:*

- *for every $a \in \mathbb{N}_0$ satisfying $2a = k$, a subrepresentation isomorphic to* `triv` *with a basis*

$$\{\overline{\sigma_2}^a\};$$

- *for every $a, b \in \mathbb{N}_0$ satisfying $2a + 3b = k$, $b > 0$, an indecomposable extension of two* *copies of* `triv`*, with a basis*

$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b,\ \overline{\sigma_2}^a\overline{\sigma_2}^{b-1}(\overline{x_1}^3+\overline{x_1}^2\overline{x_2}+\overline{x_2}^3)\};$$

- *for every $a, b \in \mathbb{N}_0$ satisfying $2a+3b=k-1$, a subrepresentation isomorphic to* `stand` *with a basis*
$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_1}, \overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_2}\};$$
- *for every $a, b \in \mathbb{N}_0$ satisfying $2a+3b=k-2$, a subrepresentation isomorphic to* `stand` *with a basis*
$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_1}^2, \overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_2}^2\}.$$

Proof. It is straightforward to check that for every $a, b \in \mathbb{N}_0$ satisfying $2a+3b=k$ the space spanned by $\overline{\sigma_2}^a\overline{\sigma_3}^b$ is a subrepresentation isomorphic to `triv`. When $b>0$, it is extended by the 1-dimensional space with a basis $\overline{\sigma_2}^a\overline{\sigma_3}^{b-1}(\overline{x_1}^3+\overline{x_1}^2\overline{x_2}+\overline{x_2}^3)$, as

$$\begin{aligned} s_1.(\overline{\sigma_2}^a\overline{\sigma_3}^{b-1}(\overline{x_1}^3+\overline{x_1}^2\overline{x_2}+\overline{x_2}^3)) &= \overline{\sigma_2}^a\overline{\sigma_3}^{b-1}(\overline{x_1}^3+\overline{x_1}^2\overline{x_2}+\overline{x_2}^3)+\overline{\sigma_2}^a\overline{\sigma_3}^b \\ s_2.(\overline{\sigma_2}^a\overline{\sigma_3}^{b-1}(\overline{x_1}^3+\overline{x_1}^2\overline{x_2}+\overline{x_2}^3)) &= \overline{\sigma_2}^a\overline{\sigma_3}^{b-1}(\overline{x_1}^3+\overline{x_1}^2\overline{x_3}+\overline{x_3}^3) \\ &= \overline{\sigma_2}^a\overline{\sigma_3}^{b-1}(\overline{x_1}^3+\overline{x_1}^2\overline{x_2}+\overline{x_2}^3)+\overline{\sigma_2}^a\overline{\sigma_3}^b. \end{aligned}$$

It is also straighforward to check that the space spanned by $\{\overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_1}, \overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_2}\}$ is a subrepresentation isomorphic to `stand` via the isomorphism $\overline{x_i} \mapsto \overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_i}$ and that the space spanned by $\{\overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_1}^2, \overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_2}^2\}$ is a subrepresentation isomorphic to `stand` via the isomorphism $\overline{x_i} \mapsto \overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_i}^2$.

The set

$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b,\ \overline{\sigma_2}^a\overline{\sigma_2}^{b-1}(\overline{x_1}^3+\overline{x_1}^2\overline{x_2}+\overline{x_2}^3) \mid 2a+3b=k\}$$

is linearly independent, and so is the set

$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_1},\ \overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_2} \mid 2a+3b=k-1\} \cup \{\overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_1}^2,\ \overline{\sigma_2}^a\overline{\sigma_3}^b\overline{x_2}^2 \mid 2a+3b=k-2\}.$$

The central element $(123)+(132) \in \Bbbk[S_3]$ acts on the span of the first of these sets by 0 and on the second of them by 1. Since $(123)+(132)$ and $e-(123)-(132)$ act as projections on the span of their union, and this allows us to conclude that the union of these sets is linearly independent as well. To show that it is a basis of $S^k(\mathfrak{h}^*)$, and thus conclude that $S(\mathfrak{h}^*)$ is indeed a direct sum as stated in the theorem, it now suffices to show that the number of elements in this set and compare it to $\dim S^k(\mathfrak{h}^*) = k+1$. This is the exact same calculation as in the proof of Theorem 7.2.2. □

Next, we will need an analogue of Lemma 7.2.3. The `sign` representation is isomorphic to `triv` in characteristic 2 and taking tensor products with `triv` is tautological, so the only representation to decompose is the tensor square of `stand`. This representation turns out to not be completely reducible in characteristic 2.

Lemma 7.2.8. *Let $p = 2$. As an S_3 representation,* $\texttt{stand} \otimes \texttt{stand}$ *is a direct sum of the following indecomposable subrepresentations:*

- *an indecomposable extension of two copies of* `triv`*, with a basis*

$$\{\overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2},\ \overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2}\},$$

  with $\overline{x_1} \otimes \overline{x_2} + \overline{x_2} \otimes \overline{x_1}$ spanning a subrepresentation of this extension;

- *a subrepresentation isomorphic to* `stand` *with a basis*

$$\{\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2},\ \overline{x_2} \otimes \overline{x_2} + \overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2}\},$$

  with the isomorphism given by $\overline{x_i} \mapsto \overline{x_i} \otimes \overline{x_i} + \overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2}$.

Proof. To show that the preceding bases are in a direct sum with each other, we write their elements in a matrix with respect to the basis $\{\overline{x_1} \otimes \overline{x_1},\ \overline{x_2} \otimes \overline{x_1},\ \overline{x_1} \otimes \overline{x_2},\ \overline{x_2} \otimes \overline{x_2}\}$ of $\texttt{stand} \otimes \texttt{stand}$.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

This matrix has determinant 1 and is therefore full rank. Hence the bases are linearly independent and in a direct sum. To show that they span S_3 subrepresentations, we consider the action of the generators (12) and (23).

$$\begin{aligned}
(12).(\overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2}) &= \overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2} \\
(23).(\overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2}) &= \overline{x_3} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_3} \\
&= (\overline{x_1} + \overline{x_2}) \otimes \overline{x_1} + \overline{x_1} \otimes (\overline{x_1} + \overline{x_2}) \\
&= \overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2}
\end{aligned}$$

This shows that $\overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2}$ spans a copy of `triv`. Now,

$$\begin{aligned}
(12).(\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2}) &= \overline{x_2} \otimes \overline{x_2} + \overline{x_1} \otimes \overline{x_2} + \overline{x_1} \otimes \overline{x_1} \\
&= \overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2} \\
&\quad + \overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2} \\
&= \overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2} \pmod{\overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2}}
\end{aligned}$$

$$\begin{aligned}
(23).(\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2}) &= \overline{x_1} \otimes \overline{x_1} + \overline{x_3} \otimes \overline{x_1} + \overline{x_3} \otimes \overline{x_3} \\
&= \overline{x_1} \otimes \overline{x_1} + (\overline{x_1} + \overline{x_2}) \otimes \overline{x_1} + (\overline{x_1} + \overline{x_2}) \otimes (\overline{x_1} + \overline{x_2}) \\
&= \overline{x_1} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2} + \overline{x_2} \otimes \overline{x_2} \\
&= \overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2} \\
&\quad + \overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2} \\
&= \overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2} \pmod{\overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2}}.
\end{aligned}$$

This shows that $\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2}$ spans a copy of `triv` modulo the previous copy of `triv`. To show that this is an indecomposable extension, consider the matrix forms for the action of (12) and (23) with respect to the basis.

$$[(12)] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$[(23)] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

These matrices are non-diagonalisable, therefore basis consists of an indecomposable extension of `triv` by `triv`.

The action of (12) and (23) on the second basis is easily calculated in matrix form as

$$[(12)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$[(23)] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

which exactly shows that they span a copy of `stand`. □

Now by putting together Theorem 7.2.7 and Lemma 7.2.8 we immediately get the following theorem.

Theorem 7.2.9. *In characteristic $p = 2$, $M^k_{t,c}(\texttt{stand}) \cong S^k(\mathfrak{h}^*) \otimes \texttt{stand}$ is a direct sum of the following indecomposable S_3 representations:*

- *for every $a \in \mathbb{N}_0$ satisfying $2a = k$, a subrepresentation isomorphic to* `stand` *with a basis*
$$\{\overline{\sigma_2}^a \otimes \overline{x_1},\ \overline{\sigma_2}^a \otimes \overline{x_2}\};$$
- *for every $a, b \in \mathbb{N}_0$ satisfying $2a + 3b = k$, $b > 0$, a direct sum of two copies of* `stand`*, with a basis*
$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b \otimes \overline{x_1},\ \overline{\sigma_2}^a\overline{\sigma_3}^b \otimes \overline{x_2}\}$$

of one subrepresentation and a basis

$$\{\overline{\sigma_2}^a\overline{\sigma_2}^{b-1}\big((\overline{x_1}^3+\overline{x_1}^2\overline{x_2}+\overline{x_2}^3)\otimes\overline{x_1}+\overline{\sigma_2}\otimes\overline{x_2}\big),\\ \overline{\sigma_2}^a\overline{\sigma_2}^{b-1}\big((\overline{x_1}^3+\overline{x_1}^2\overline{x_2}+\overline{x_2}^3)\otimes\overline{x_2}+\overline{\sigma_3}\otimes(\overline{x_1}+\overline{x_2})\big)\}$$

of the other;

- *for every $a, b \in \mathbb{N}_0$ satisfying $2a+3b = k-1$, an indecomposable extension of two copies of* `triv`*, with a basis*
$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b(\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1})\}$$
of the subrepresentation isomorphic to `triv` *and a basis*
$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b(\overline{x_1}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_2})\}$$
of the quotient isomorphic to `triv`*;*

- *for every $a, b \in \mathbb{N}_0$ satisfying $2a+3b = k-1$, a subrepresentation isomorphic to* `stand` *with a basis*
$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b(\overline{x_1}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_1}+\overline{x_1}\otimes\overline{x_2}),\ \overline{\sigma_2}^a\overline{\sigma_3}^b(\overline{x_2}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1}+\overline{x_1}\otimes\overline{x_2})\};$$

- *for every $a, b \in \mathbb{N}_0$ satisfying $2a+3b = k-2$, an indecomposable extension of two copies of* `triv`*, with a basis*
$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b(\overline{x_1}^2\otimes\overline{x_2}+\overline{x_2}^2\otimes\overline{x_1})\}$$
of the subrepresentation isomorphic to `triv` *and a basis*
$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b(\overline{x_1}^2\otimes\overline{x_1}+\overline{x_2}^2\otimes\overline{x_1}+\overline{x_2}^2\otimes\overline{x_2})\}$$
of the quotient isomorphic to `triv`*;*

- *for every $a, b \in \mathbb{N}_0$ satisfying $2a+3b = k-2$, a subrepresentation isomorphic to* `stand` *with a basis*
$$\{\overline{\sigma_2}^a\overline{\sigma_3}^b(\overline{x_1}^2\otimes\overline{x_1}+\overline{x_2}^2\otimes\overline{x_1}+\overline{x_1}^2\otimes\overline{x_2}),\ \overline{\sigma_2}^a\overline{\sigma_3}^b(\overline{x_2}^2\otimes\overline{x_2}+\overline{x_2}^2\otimes\overline{x_1}+\overline{x_1}^2\otimes\overline{x_2})\}.$$

Corollary 7.2.10. *Assume $p = 2$.*

1. *The character of the graded S_3 representation $S(\mathfrak{h}^*)$ is*
$$\chi_{S(\mathfrak{h}^*)}(z) = \frac{1}{(1-z^2)(1-z^3)}\left((1+z^3)[\texttt{triv}]+(z+z^2)[\texttt{stand}]\right).$$

2. The characters of Verma modules for $H_{t,c}(S_3, \mathfrak{h})$ are given by

$$\chi_{M_{t,c}(\texttt{triv})}(z) = \chi_{S(\mathfrak{h}^*)}(z)$$
$$\chi_{M_{t,c}(\texttt{stand})}(z) = \frac{1}{(1-z^2)(1-z^3)}\left((1+z+z^2+z^3)[\texttt{stand}] + 2(z+z^2)[\texttt{triv}]\right).$$

3. The characters of baby Verma modules for $H_{t,c}(S_3, \mathfrak{h})$ are given by

$$\chi_{N_{0,c}(\tau)}(z) = \chi_{M_{t,c}(\tau)}(z)(1-z^2)(1-z^3).$$
$$\chi_{N_{1,c}(\tau)}(z) = \chi_{M_{t,c}(\tau)}(z)(1-z^{2p})(1-z^{3p}).$$

Proof. We follow the proof of Corollary 7.2.5. Using the decomposition from Theorem 7.2.7 we get

$$\begin{aligned}
&\chi_{S(\mathfrak{h}^*)}(z) = \\
&= \sum_a z^{2a}[\texttt{triv}] + \sum_{\substack{a,b\\ b>0}} z^{2a+3b}\big([\texttt{triv}] + [\texttt{triv}]\big) + \sum_{a,b} z^{2a+3b+1}[\texttt{stand}] + \sum_{a,b} z^{2a+3b+2}[\texttt{stand}] \\
&= \frac{1}{(1-z^2)}[\texttt{triv}] + \frac{1}{(1-z^2)(1-z^3)}\big(z^3([\texttt{triv}] + [\texttt{triv}]) + (z+z^2)[\texttt{stand}]\big) \\
&= \frac{1}{(1-z^2)(1-z^3)}\big((1-z^3)[\texttt{triv}] + 2z^3[\texttt{triv}] + (z+z^2)[\texttt{stand}]\big) \\
&= \frac{1}{(1-z^2)(1-z^3)}\big((1+z^3)[\texttt{triv}] + (z+z^2)[\texttt{stand}]\big)
\end{aligned}$$

The characters of Verma modules then follow from $M_{t,c}(\tau) = S(\mathfrak{h}^*) \otimes \tau$ and Lemma 7.2.8, and the characters of baby Verma modules follow from the fact that $N_{t,c}(\tau)$ is a quotient of $M_{t,c}(\tau)$ by $\overline{\sigma_i} \otimes \tau$ (when $t = 0$) or by $\overline{\sigma_i}^p \otimes \tau$ (when $t = 1$).

□

Chapter 8

Irreducible Representations of $H_{t,c}(S_3, \mathfrak{h})$ in Characteristic 2

Representation theory of S_3 over an algebraically closed field $\Bbbk$ of characteristic 2 is not semisimple, and the irreducible representations are the trivial representation `triv` and the standard representation `stand`; the sign representation `sign` is isomorphic to `triv`.

Although the permutation representation V of S_3 is semisimple in characteristic 2 and splits as $V \cong \mathfrak{h} \oplus \mathtt{triv}$, we will nevertheless consider $\mathfrak{h}^*$ as a quotient of V^* rather than realise it as a submodule. For more details on the structure of these representations, see Section 3.1 and Section 3.2. We work with the spanning set $\{\overline{x_1}, \overline{x_2}, \overline{x_3}\}$ satisfying $\overline{x_1} + \overline{x_2} + \overline{x_3} = 0$, and often choose $\overline{x_1}, \overline{x_2}$ as a basis of $\mathfrak{h}^*$. We do this to match the conventions in the literature in this case.

The aim of this section is to prove the following theorem.

Theorem 8.0.1. *The characters and Hilbert polynomials of the irreducible representation $L_{t,c}(\tau)$ of the rational Cherednik algebra $H_{t,c}(S_3)$ over an algebraically closed field of characteristic 2, for any t, c and τ, are given by the following tables.*

Characters:

$p = 2$	$\tau = \mathtt{triv}$
$t = 0,\ c = 0$	$[\mathtt{triv}]$
$t = 0,\ c \neq 0$	$[\mathtt{triv}] + [\mathtt{stand}](z + z^2) + [\mathtt{triv}]z^3$
$t = 1,\ c \notin \mathbb{F}_2$	$\big([\mathtt{triv}] + [\mathtt{stand}]z + [\mathtt{triv}]z^2\big)\big([\mathtt{triv}] + [\mathtt{stand}]z^2 + [\mathtt{stand}]z^4 + [\mathtt{triv}]z^6\big)$
$t = 1,\ c = 0$	$[\mathtt{triv}] + [\mathtt{stand}]z + [\mathtt{triv}]z^2$
$t = 1,\ c = 1$	$[\mathtt{triv}]$

$\boldsymbol{p = 2}$	$\tau =$ `stand`
$t = 0,\ c = 0$	$[\texttt{stand}]$
$t = 0,\ c \neq 0$	$[\texttt{stand}] + ([\texttt{triv}] + [\texttt{sign}])z + [\texttt{stand}]z^2$
$t = 1,\ c \notin \mathbb{F}_2$	$[\texttt{stand}](1 + z + z^2 + 2z^3 + z^4 + z^5 + z^6) + 2[\texttt{triv}](z + z^2 + z^4 + z^5)$
$t = 1,\ c = 0$	$[\texttt{stand}] + \big([\texttt{stand}] + 2[\texttt{triv}]\big)z + [\texttt{stand}]z^2$
$t = 1,\ c = 1$	$[\texttt{stand}](1 + z + z^2 + 2z^3 + z^4 + z^5 + z^6) + [\texttt{triv}](z + 2z^2 + 2z^4 + z^5)$

Hilbert polynomials:

$\boldsymbol{p = 2}$	$\tau =$ `triv`	$\tau =$ `stand`
$t = 0,\ c = 0$	1	2
$t = 0,\ c \neq 0$	$1 + 2z + 2z^2 + z^3$ *[CaKa21], Thm 2.11*	$2 + 2z + 2z^2$
$t = 1,\ c \notin \mathbb{F}_2$	$\dfrac{(1 - z^4)(1 - z^6)}{(1 - z)^2}$ *[CaKa21], Thm 3.17*	$\dfrac{2(1 - z^2)(1 - z^6)}{(1 - z)^2}$
$t = 1,\ c = 0$	$(1 + z)^2$	$2 + 4z + 2z^2$
$t = 1,\ c = 1$	1 *[Li14], Thm. 3.2*	$\dfrac{2 - z - z^3 - z^5 - z^7 + 2z^8}{(1 - z)^2}$

In all cases, the singular vectors are known explicitly and they are given in the following lemmas and propositions.

Proof. The irreducible representation $L_{t,c}(\texttt{triv})$ is described in the following lemmas and propositions:

- for $t = 0,\ c = 0$ in Proposition 2.6.11 or Proposition 4.1.4;
- for $t = 0,\ c \neq 0$ in Lemma 8.1.2;
- for $t = 1,\ c \neq 0, 1$ in Lemma 8.2.2;
- for $t = 1,\ c = 0$ in Lemma 2.6.13;
- for $t = 1,\ c = 1$ in Proposition 4.1.4.

The irreducible representation $L_{t,c}(\texttt{stand})$ is described in the following lemmas and propositions:

- for $t = 0,\ c = 0$ in Proposition 2.6.11;
- for $t = 0, c \neq 0$ in Lemma 8.3.3;

- for $t = 1$, $c \neq 0, 1$ in Lemma 8.4.7 and Lemma 8.4.12;
- for $t = 1$, $c = 0$ in Lemma 2.6.13;
- for $t = 1$, $c = 1$ in Lemma 8.5.1.

□

8.1 The irreducible representation $L_{0,c}(\mathtt{triv})$ in characteristic 2 for $c \neq 0$

Theorem 8.1.1 ([CaKa21], Theorem 2.11)**.** *Let* $\Bbbk$ *be an algebraically closed field of characteristic* 2*, let* n *be an arbitrary odd integer, the parameter* $t = 0$ *and the parameter* $c \in \Bbbk$ *generic. The singular vectors in* $M_{0,c}(\mathtt{triv})$ *which generate the maximal proper graded submodule are* $\overline{x_i}^2 + \overline{x_i x_j} + \overline{x_j}^2, \overline{x_i x_j x_k}, \quad i < j < k < n$*. The Hilbert polynomial of the irreducible representation* $L_{0,c}(\mathtt{triv})$ *of* $H_{0,c}(S_n, \mathfrak{h})$ *equals*

$$Hilb_{L_{0,c}(\mathtt{triv})}(z) = (1 + z)(1 + (n - 2)z + z^2).$$

Let us reprove a special case of this theorem, while also calculating the character of the representation $L_{0,c}(\mathtt{triv})$, and clarifying that "generic c" in this case means $c \neq 0$. The special case we will prove is the following lemma. The proof is following [CaKa21], simplifying where possible.

Lemma 8.1.2. *Let* $\Bbbk$ *be an algebraically closed field of characteristic* 2*, and let the values of the parameters be* $t = 0$ *and* $c \neq 0$*. The irreducible representation* $L_{0,c}(\mathtt{triv})$ *of* $H_{0,c}(S_3, \mathfrak{h})$ *is equal to the baby Verma module* $N_{0,c}(\mathtt{triv})$*, with the character*

$$\chi_{L_{0,c}(\mathtt{triv})}(z) = [\mathtt{triv}] + [\mathtt{stand}]z + [\mathtt{stand}]z^2 + [\mathtt{triv}]z^3$$

and the Hilbert polynomial

$$Hilb_{L_{0,c}(\mathtt{triv})}(z) = (1 + z)(1 + z + z^2) = 1 + 2z + 2z^2 + z^3.$$

Proof. When $n = 3$, the elementary symmetric polynomials in $S(\mathfrak{h}^*)$ are

$$\overline{\sigma_2} = \overline{x_1 x_2 + x_1 x_3 + x_2 x_3} = \overline{x_1 x_2} + (\overline{x_1} + \overline{x_2})^2 = \overline{x_1}^2 + \overline{x_1 x_2} + \overline{x_2}^2$$

and

$$\overline{\sigma_3} = \overline{x_1 x_2 x_3}.$$

Note that these are exactly the vectors which Cai-Kalinov found above. The fact they are symmetric immediately shows they are singular at $t = 0$. The quotient of $M_{0,c}(\mathtt{triv})$ by the

submodule generated by these vectors is the baby Verma module $N_{0,c}(\texttt{triv})$, which has the above stated character and Hilbert polynomial (see Example 4.1.1).
It remains to show that whenever $c \neq 0$ this baby Verma module is irreducible.
Let $a_1, a_2 \in \Bbbk$, let $i = 1$ or $i = 2$, and calculate

$$D_{y_i - y_3}(a_1\overline{x_1} + a_2\overline{x_2}) = ca_i.$$

This shows that whenever $c \neq 0$, there are no singular vectors in degree 1 of $N_{0,c}(\texttt{triv})$.
In degree 2, a basis of $M^2_{0,c}(\texttt{triv}) \cong S^2(\mathfrak{h}^*)$ is $\{\overline{x_1}^2, \overline{x_1x_2}, \overline{x_2}^2\}$, and $\overline{\sigma_2}$ is their sum. After taking the quotient by $\overline{\sigma_2}$, we can choose $\{\overline{x_1}^2, \overline{x_2}^2\}$ as a basis of $N^2_{0,c}(\texttt{triv})$. To look for singular vectors in $N^2_{0,c}(\texttt{triv})$ we calculate, for $a_1, a_2 \in \Bbbk$

$$\begin{aligned} D_{y_1-y_3}(a_1\overline{x_1}^2 + a_2\overline{x_2}^2) &= -c\frac{\big(1-(12)\big)}{\overline{x_1}-\overline{x_2}}(a_1\overline{x_1}^2 + a_2\overline{x_2}^2) - c\frac{\big(1-(23)\big)}{\overline{x_2}-\overline{x_3}}(a_1\overline{x_1}^2 + a_2\overline{x_2}^2) \\ &= c\frac{(a_1+a_2)(\overline{x_1}^2 - \overline{x_2}^2)}{\overline{x_1}-\overline{x_2}} + c\frac{a_2\overline{x_2}^2 - a_2\overline{x_3}^2}{\overline{x_2}-\overline{x_3}} \\ &= c\big(a_1\overline{x_1} + (a_1+a_2)\overline{x_2}\big). \end{aligned}$$

If $c \neq 0$, the only vectors of the form $a_1\overline{x_1}^2 + a_2\overline{x_2}^2$ in the kernel of $D_{y_1-y_3}$ have $a_1 = 0$ and $a_1 + a_2 = 0$, which shows that there are no singular vectors in $N^2_{0,c}(\texttt{triv})$.
Finally, a basis of $N^3_{0,c}(\texttt{triv})$ is $\{\overline{x_1}^2\overline{x_2}\}$, and we have

$$\begin{aligned} D_{y_1-y_3}(\overline{x_1}^2\overline{x_2}) &= -c\frac{\overline{x_1}^2\overline{x_2} - \overline{x_1x_2}^2}{\overline{x_1}-\overline{x_2}} - c\frac{\overline{x_1}^2\overline{x_2} - \overline{x_1}^2\overline{x_3}}{\overline{x_2}-\overline{x_3}} \\ &= c\left(\overline{x_1x_2} + \overline{x_1}^2\right), \end{aligned}$$

which is not zero in $N^2_{0,c}(\texttt{triv})$ when $c \neq 0$.
This shows that when $c \neq 0$, $N_{0,c}(\texttt{triv})$ is irreducible. □

8.2 The irreducible representation $L_{1,c}(\texttt{triv})$ in characteristic 2 for generic c

Theorem 8.2.1 ([CaKa21], Theorem 3.17)**.** *Let $\Bbbk$ be an algebraically closed field of characteristic 2, let n be an arbitrary odd integer, the parameter $t = 1$ and the parameter $c \in \Bbbk$ transcendental over $\mathbb{F}_2$. The Hilbert polynomial of the irreducible representation $L_{1,c}(\texttt{triv})$ of $H_{1,c}(S_n)$ equals*

$$Hilb_{L_{1,c}(\texttt{triv})}(z) = (1+z)^{n-1}(1 + (n-1)z^2 + (n-1)z^4 + z^6).$$

So, in this case the reduced Hilbert polynomial (in the sense of [BaCh13a], Proposition 3.4) of $L_{1,c}(\texttt{triv})$ equals the Hilbert polynomial of $L_{0,c}(\texttt{triv})$.

Again, let us reprove a special case of this theorem, while also calculating the character of the representation $L_{1,c}(\texttt{triv})$, and clarifying that "generic c" in this case means $c \notin \mathbb{F}_2$. (note: alternatively, this follows from Theorem 4.1.3). The special case we will prove is the following lemma. The proof is using [CaKa21] but can be greatly simplified in this case.

Lemma 8.2.2. *Let $\Bbbk$ be an algebraically closed field of characteristic 2, and let the values of the parameters be $t = 1$ and $c \neq 0, 1$. The irreducible representation $L_{1,c}(\texttt{triv})$ of $H_{1,c}(S_3, \mathfrak{h})$ is equal to the baby Verma module $N_{1,c}(\texttt{triv})$, with the character*

$$\chi_{L_{1,c}(\texttt{triv})}(z) = \big([\texttt{triv}] + [\texttt{stand}]z + [\texttt{triv}]z^2\big)\big([\texttt{triv}] + [\texttt{stand}]z^2 + [\texttt{stand}]z^4 + [\texttt{triv}]z^6\big)$$

and the Hilbert polynomial

$$Hilb_{L_{1,c}(\texttt{triv})}(z) = (1+z)^2(1+2z^2+2z^4+z^6).$$

Proof. The baby Verma module $N_{1,c}(\texttt{triv})$ is the quotient of $M_{1,c}(\texttt{triv})$ by the graded submodule generated by $\overline{\sigma_2}^2$ and $\overline{\sigma_3}^2$. The *reduced module* is the quotient of $M_{1,c}(\texttt{triv}) \cong S(\mathfrak{h}^*)$ by the submodule generated by $\overline{\sigma_2}$ and $\overline{\sigma_3}$ without p^{th} powers, as described in [BaCh13a], Proposition 3.4. Its character is called the *reduced character*, denoted $H(z)$, and is equal to

$$\begin{aligned} H(z) &= \chi_{S(\mathfrak{h}^*)/(\sigma_2 S(\mathfrak{h}^*)+\sigma_3 S(\mathfrak{h}^*))} \\ &= \chi_{S(\mathfrak{h}^*)}(1-z^2)(1-z^3) \\ &= [\texttt{triv}] + [\texttt{stand}]z + [\texttt{stand}]z^2 + [\texttt{triv}]z^3. \end{aligned}$$

Again by [BaCh13a], Proposition 3.4, the character of $N_{1,c}(\texttt{triv})$ is

$$\chi_{N_{1,c}(\texttt{triv})}(z) = \chi_{S^{(2)}(\mathfrak{h}^*)} \cdot H(z^2).$$

where $S^{(p)}(\mathfrak{h}^*)$ is the quotient of $S(\mathfrak{h}^*)$ by the ideal generated by $\{x^p \mid x \in \mathfrak{h}^*\}$. Using

$$\chi_{S^{(2)}(\mathfrak{h}^*)} = [\texttt{triv}] + [\texttt{stand}]z + [\texttt{triv}]z^2,$$

the character of $N_{1,c}(\texttt{triv})$ can be calculated as

$$\begin{aligned} \chi_{N_{1,c}(\texttt{triv})}(z) &= \chi_{S^{(2)}(\mathfrak{h}^*)} \cdot H(z^2) \\ &= \big([\texttt{triv}] + [\texttt{stand}]z + [\texttt{triv}]z^2\big)\big([\texttt{triv}] + [\texttt{stand}]z^2 + [\texttt{stand}]z^4 + [\texttt{triv}]z^6\big) \end{aligned}$$

It remains to show that when $c \neq 0, 1$, the baby Verma module $N_{1,c}(\texttt{triv})$ is irreducible. Let us start with the computation that will show that any submodule of $N_{1,c}(\texttt{triv})$ containing the top degree also contains the bottom degree. We choose the basis of monomials $\{\overline{x_1}^i \overline{x_2}^j\}$

of $S(\mathfrak{h}^*) \cong M_{1,c}(\texttt{triv})$. After taking the quotient by

$$\begin{aligned}
\overline{\sigma_2}^2 &= \overline{x_1}^4 + \overline{x_1}^2\overline{x_2}^2 + \overline{x_2}^4 \\
\overline{\sigma_3}^2 &= \overline{x_1}^2\overline{x_2}^2\overline{x_3}^2 \\
&= \overline{x_1}^4\overline{x_2}^2 + \overline{x_1}^2\overline{x_2}^4 \\
&= \overline{x_1}^6 + \overline{x_1}^2\sigma_2^2
\end{aligned}$$

we can choose the basis $\{\overline{x_1}^i\overline{x_2}^j \mid i < 6,\ j < 4\}$ of $N_{1,c}(\texttt{triv})$. In particular the top degree $N^8_{1,c}(\texttt{triv})$ is 1-dimensional and spanned by $\overline{x_1}^5\overline{x_2}^3$.
Assume U is a $H_{1,c}(S_3,\mathfrak{h})$-submodule of $N_{1,c}(\texttt{triv})$ containing the top degree, so $\overline{x_1}^5\overline{x_2}^3 \in U$. Then U also contains $D^3_{y_1-y_2}D^5_{y_1-y_3}(\overline{x_1}^5\overline{x_2}^3)$. Let us calculate this vector. In the following computation we use $\overline{x_1}+\overline{x_2}+\overline{x_3}=0$ and $\overline{\sigma_2}^2=0$, $\overline{\sigma_3}^2=0$ in $N_{1,c}(\texttt{triv})$.

$$\begin{aligned}
D_{y_1-y_3}(\overline{x_1}^5\overline{x_2}^3) &= \partial_{y_1-y_3}(\overline{x_1}^5\overline{x_2}^3) - c\left(\frac{\overline{x_1}^5\overline{x_2}^3 - \overline{x_1}^3\overline{x_2}^5}{\overline{x_1}-\overline{x_2}} + \frac{\overline{x_1}^5\overline{x_2}^3 - \overline{x_1}^5\overline{x_3}^3}{\overline{x_2}-\overline{x_3}}\right) \\
&= 5\overline{x_1}^4\overline{x_2}^3 + c\left(\overline{x_1}^4\overline{x_2}^3 + \overline{x_1}^3\overline{x_2}^4 + \overline{x_1}^5\overline{x_2}^2 + \overline{x_1}^5\overline{x_2}\overline{x_3} + \overline{x_1}^5\overline{x_3}^2\right) \\
&= \overline{x_1}^4\overline{x_2}^3 + c\ \left(\overline{x_1}^4\overline{x_2}^3 + \overline{x_1}^3\overline{x_2}^4 + \overline{x_1}^5\overline{x_2}^2 + \overline{x_1}^6\overline{x_2} + \overline{x_1}^5\overline{x_2}^2 + \overline{x_1}^7 + \overline{x_1}^5\overline{x_2}^2\right) \\
&= \overline{x_1}^4\overline{x_2}^3 + c\left(\overline{x_1}^4\overline{x_2}^3 + \overline{x_1}^3\overline{x_2}^4 + \overline{x_1}^5\overline{x_2}^2\right) \\
&= (1+c)\overline{x_1}^4\overline{x_2}^3.
\end{aligned}$$

Next,

$$\begin{aligned}
D^2_{y_1-y_3}(\overline{x_1}^5\overline{x_2}^3) &= D_{y_1-y_3}((1+c)\overline{x_1}^4\overline{x_2}^3) \\
&= (1+c)\left(4\overline{x_1}^3\overline{x_2}^3 - c\left(\frac{\overline{x_1}^4\overline{x_2}^3 - \overline{x_1}^3\overline{x_2}^4}{\overline{x_1}-\overline{x_2}} + \frac{\overline{x_1}^4\overline{x_2}^3 - \overline{x_1}^4\overline{x_3}^3}{\overline{x_2}-\overline{x_3}}\right)\right) \\
&= (1+c)c\left(\overline{x_1}^3\overline{x_2}^3 + \overline{x_1}^4(\overline{x_2}^2 + \overline{x_2}\overline{x_3} + \overline{x_3}^2)\right) \\
&= (1+c)c\left(\overline{x_1}^3\overline{x_2}^3 + \overline{x_1}^4(\overline{x_2}^2 + \overline{x_2}\overline{x_1} + \overline{x_1}^2)\right) \\
&= (1+c)c\left(\overline{x_1}^3\overline{x_2}^3 + \overline{x_1}^4\overline{x_2}^2 + \overline{x_1}^5\overline{x_2} + \overline{x_1}^6\right) \\
&= (1+c)c\left(\overline{x_1}^5\overline{x_2} + \overline{x_1}^4\overline{x_2}^2 + \overline{x_1}^3\overline{x_2}^3\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
D^3_{y_1-y_3}(\overline{x_1}^5\overline{x_2}^3) &= (1+c)cD_{y_1-y_3}\left(\overline{x_1}^5\overline{x_2} + \overline{x_1}^4\overline{x_2}^2 + \overline{x_1}^3\overline{x_2}^3\right) \\
&= (1+c)c\Big(\overline{x_1}^4\overline{x_2} + \overline{x_1}^2\overline{x_2}^3 + c\Big(\overline{x_1}^4\overline{x_2} + \overline{x_1}^3\overline{x_2}^2 + \overline{x_1}^2\overline{x_2}^3 + \overline{x_1}\overline{x_2}^4 + \overline{x_1}^5 \\
&\qquad + \overline{x_1}^3\overline{x_2}^2 + \overline{x_1}^2\overline{x_2}^3 + \overline{x_1}^4\left(\overline{x_2}+\overline{x_3}\right) \\
&\qquad + \overline{x_1}^3\left(\overline{x_2}^2 + \overline{x_2}\overline{x_3} + \overline{x_3}^2\right)\Big)\Big) \\
&= (1+c)c\left(\overline{x_1}^4\overline{x_2} + \overline{x_1}^2\overline{x_2}^3 + c(\overline{x_1}^4\overline{x_2} + \overline{x_1}\overline{x_2}^4 + \overline{x_1}^5 + \overline{x_1}^4\overline{x_2} + \overline{x_1}^3\overline{x_2}^2)\right)
\end{aligned}$$

$$= (1+c)c\big(\overline{x_1}^4\overline{x_2} + \overline{x_1}^2\overline{x_2}^3 + c(\overline{x_1}^5 + \overline{x_1}^3\overline{x_2}^2 + \overline{x_1}\overline{x_2}^4)\big)$$
$$= (1+c)c\left(\overline{x_1}^4\overline{x_2} + \overline{x_1}^2\overline{x_2}^3\right).$$

Next,

$$\begin{aligned}
D^4_{y_1-y_3}(\overline{x_1}^5\overline{x_2}^3) &= (1+c)cD_{y_1-y_3}\left(\overline{x_1}^4\overline{x_2} + \overline{x_1}^2\overline{x_2}^3\right) \\
&= (1+c)c^2\left(\overline{x_1}^3\overline{x_2} + \overline{x_1}^2\overline{x_2}^2 + \overline{x_1}\overline{x_2}^3 + \overline{x_1}^4 + \overline{x_1}^2\overline{x_2}^2 + \overline{x_1}^2(\overline{x_2}^2 + \overline{x_2}\overline{x_3} + \overline{x_3}^2)\right) \\
&= (1+c)c^2\left(\overline{x_1}^3\overline{x_2} + \overline{x_1}\overline{x_2}^3 + \overline{x_1}^4 + \overline{x_1}^4 + \overline{x_1}^3\overline{x_2} + \overline{x_1}^2\overline{x_2}^2\right) \\
&= (1+c)c^2\left(\overline{x_1}^2\overline{x_2}^2 + \overline{x_1}\overline{x_2}^3\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
D^5_{y_1-y_3}(\overline{x_1}^5\overline{x_2}^3) &= (1+c)c^2D_{y_1-y_3}\left(\overline{x_1}^2\overline{x_2}^2 + \overline{x_1}\overline{x_2}^3\right) \\
&= (1+c)c^2\left(\overline{x_2}^3 - c\left(\overline{x_1}^2(\overline{x_2} + \overline{x_3}) + \overline{x_1}^2\overline{x_2} + \overline{x_1}\overline{x_2}^2 + \overline{x_1}(\overline{x_2}^2 + \overline{x_2}\overline{x_3} + \overline{x_3}^2)\right)\right) \\
&= (1+c)c^2\left(\overline{x_2}^3 - c\left(\overline{x_1}^3 + \overline{x_1}^2\overline{x_2} + \overline{x_1}\overline{x_2}^2 + \overline{x_1}^3 + \overline{x_1}^2\overline{x_2} + \overline{x_1}\overline{x_2}^2\right)\right) \\
&= (1+c)c^2\overline{x_2}^3.
\end{aligned}$$

Next,

$$\begin{aligned}
D_{y_1-y_2}D^5_{y_1-y_3}(\overline{x_1}^5\overline{x_2}^3) &= (1+c)c^2D_{y_1-y_2}(\overline{x_2}^3) \\
&= (1+c)c^2\left(-3\overline{x_2}^2 - c\left(\frac{\overline{x_2}^3 - \overline{x_2}^3}{\overline{x_1} - \overline{x_3}} - \frac{\overline{x_2}^3 - \overline{x_3}^3}{\overline{x_2} - \overline{x_3}}\right)\right) \\
&= (1+c)c^2\left(\overline{x_2}^2 + c\left(\overline{x_2}^2 + \overline{x_2}\overline{x_3} + \overline{x_3}^2\right)\right) \\
&= (1+c)c^2\left(c\overline{x_1}^2 + c\overline{x_1}\overline{x_2} + (1+c)\overline{x_2}^2\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
D^2_{y_1-y_2}D^5_{y_1-y_3}(\overline{x_1}^5\overline{x_2}^3) &= (1+c)c^2D_{y_1-y_2}\left(c\overline{x_1}^2 + c\overline{x_1}\overline{x_2} + (1+c)\overline{x_2}^2\right) \\
&= (1+c)c^2\left(c\overline{x_1} + c\overline{x_2} - c(c\overline{x_1} + c\overline{x_3} + c\overline{x_2} + c\overline{x_1} + (1+c)\overline{x_2} + (1+c)\overline{x_3})\right) \\
&= (1+c)c^2\left(c\overline{x_1} + c\overline{x_2} + c\overline{x_1}\right) \\
&= (1+c)c^3\overline{x_2}.
\end{aligned}$$

Finally,

$$D^3_{y_1-y_2}D^5_{y_1-y_3}(\overline{x_1}^5\overline{x_2}^3) = (1+c)c^3D_{y_1-y_2}(\overline{x_2}) = (1+c)^2c^3.$$

Now assume that $c \neq 0, 1$ and that $U \neq 0$ is an $H_{1,c}(S_3, \mathfrak{h})$-submodule of $N_{1,c}(\texttt{triv})$. All submodules of $N_{1,c}(\texttt{triv})$ are graded. The module $N_{1,c}(\texttt{triv}) \cong S(\mathfrak{h}^*)/(\overline{\sigma_2}^2S(\mathfrak{h}^*) + \overline{\sigma_3}^2S(\mathfrak{h}^*))$ inherits the graded algebra structure from $S(\mathfrak{h}^*)$ and is a Frobenius algebra with respect to

this structure, meaning that for every k the multiplication $N^k_{1,c}(\texttt{triv}) \times N^{8-k}_{1,c}(\texttt{triv}) \to N^8_{1,c}(\texttt{triv})$ gives a nondegenerate pairing. In particular, any nonzero $S(\mathfrak{h}^*)$-submodule of $N_{1,c}(\texttt{triv})$ contains the top degree vector $\overline{x_1}^5\overline{x_2}^3$. We conclude that $\overline{x_1}^5\overline{x_2}^3 \in U$.
The above calculation then shows that

$$\frac{1}{(1+c)^2c^3}D^3_{y_1-y_2}D^5_{y_1-y_3}(\overline{x_1}^5\overline{x_2}^3) = 1 \in U$$

and so U contains the lowest degree of $N_{1,c}(\texttt{triv})$. The baby Verma module is generated (as an $S(\mathfrak{h}^*)$-module or an $H_{1,c}(S_3,\mathfrak{h})$-module) by its lowest graded piece, so $U = N_{1,c}(\texttt{triv})$. This shows that $N_{1,c}(\texttt{triv})$ is irreducible whenever $c \neq 0,1$. □

Remark 8.2.3. An alternative proof of the above lemma is to use Theorems 4.1.3 and 8.2.1.

8.3 The irreducible representation $L_{0,c}(\texttt{stand})$ in characteristic $p \neq 3$ for generic c

In the following sections we will describe the irreducible representations of rational Cherednik algebras $H_{t,c}(S_3,\mathfrak{h})$ with lowest weight `stand`, for all values of parameters t and c. The aim is to do so over an algebraically closed field of characteristic 2, but the proofs work over an algebraically closed field of any characteristic $p \neq 3$.

We will prove the following lemma in a way which will allow us to use it in later sections, and then we will restate it in the current conventions.

Lemma 8.3.1. *Let $\Bbbk$ be a field of arbitrary characteristic and $t,c \in \Bbbk$ arbitrary parameters for the rational Cherednik algebra $H_{t,c}(S_3,\mathfrak{h})$. The matrices of the Dunkl operators $D_{y_1-y_2}$ and $D_{y_2-y_3}$ restricted to degree 1 of $M_{t,c}(\texttt{stand})$ when written with respect to the bases $\{\overline{x_1}\otimes\overline{x_1}, \overline{x_2}\otimes\overline{x_1}, \overline{x_1}\otimes\overline{x_2}, \overline{x_2}\otimes\overline{x_2}\}$ for $M^1_{t,c}(\texttt{stand})$ and $\{1\otimes\overline{x_1}, 1\otimes\overline{x_2}\}$ for $M^0_{t,c}(\texttt{stand})$ are:*

$$\left[(D_{y_1-y_2})|_{M^1_{t,c}(\texttt{stand})}\right] = \begin{array}{c} \begin{matrix} \overline{x_1}\otimes\overline{x_1} & \overline{x_2}\otimes\overline{x_1} & \overline{x_1}\otimes\overline{x_2} & \overline{x_2}\otimes\overline{x_2} \end{matrix} \\ \begin{bmatrix} t+c & -t+c & -2c & c \\ -c & 2c & t-c & -t-c \end{bmatrix} \end{array} \begin{matrix} 1\otimes\overline{x_1} \\ 1\otimes\overline{x_2} \end{matrix}$$

$$\left[(D_{y_2-y_3})|_{M^1_{t,c}(\texttt{stand})}\right] = \begin{array}{c} \begin{matrix} \overline{x_1}\otimes\overline{x_1} & \overline{x_2}\otimes\overline{x_1} & \overline{x_1}\otimes\overline{x_2} & \overline{x_2}\otimes\overline{x_2} \end{matrix} \\ \begin{bmatrix} c & t-2c & c & c \\ 2c & -c & -c & t+2c \end{bmatrix} \end{array} \begin{matrix} 1\otimes\overline{x_1} \\ 1\otimes\overline{x_2} \end{matrix}$$

Proof. The computation goes as follows:

$$D_{y_1-y_2}(\overline{x_1}\otimes\overline{x_1}) = t\partial_{y_1-y_2}(\overline{x_1})\otimes\overline{x_1} - c\,\langle y_1-y_2, \overline{x_1}-\overline{x_2}\rangle\,\frac{\overline{x_1}-(12).\overline{x_1}}{\overline{x_1}-\overline{x_2}}\otimes(12).\overline{x_1}$$

$$
\begin{aligned}
&- c\,\langle y_1 - y_2, \overline{x_1} - \overline{x_3}\rangle \frac{\overline{x_1} - (13).\overline{x_1}}{\overline{x_1} - \overline{x_3}} \otimes (13).\overline{x_1} \\
&- c\,\langle y_1 - y_2, \overline{x_2} - \overline{x_3}\rangle \frac{\overline{x_1} - (23).\overline{x_1}}{\overline{x_2} - \overline{x_3}} \otimes (23).\overline{x_1} \\
&= t \cdot 1 \otimes \overline{x_1} - 2c \cdot \frac{\overline{x_1} - \overline{x_2}}{\overline{x_1} - \overline{x_2}} \otimes \overline{x_2} - c \cdot \frac{\overline{x_1} - \overline{x_3}}{\overline{x_1} - \overline{x_3}} \otimes \overline{x_3} + c \cdot \frac{\overline{x_1} - \overline{x_1}}{\overline{x_2} - \overline{x_3}} \otimes \overline{x_1} \\
&= t \cdot 1 \otimes \overline{x_1} - 2c \cdot 1 \otimes \overline{x_2} - c \cdot 1 \otimes \overline{x_3} \\
&= (t + c) \cdot 1 \otimes \overline{x_1} - c \cdot 1 \otimes \overline{x_2}.
\end{aligned}
$$

$$
\begin{aligned}
D_{y_1-y_2}(\overline{x_2} \otimes \overline{x_1}) &= t\partial_{y_1-y_2}(\overline{x_2}) \otimes \overline{x_1} - c\,\langle y_1 - y_2, \overline{x_1} - \overline{x_2}\rangle \frac{\overline{x_2} - (12).\overline{x_2}}{\overline{x_1} - \overline{x_2}} \otimes (12).\overline{x_1} \\
&\quad - c\,\langle y_1 - y_2, \overline{x_1} - \overline{x_3}\rangle \frac{\overline{x_2} - (13).\overline{x_2}}{\overline{x_1} - \overline{x_3}} \otimes (13).\overline{x_1} \\
&\quad - c\,\langle y_1 - y_2, \overline{x_2} - \overline{x_3}\rangle \frac{\overline{x_2} - (23).\overline{x_2}}{\overline{x_2} - \overline{x_3}} \otimes (23).\overline{x_1} \\
&= -t \cdot 1 \otimes \overline{x_1} - 2c \cdot \frac{\overline{x_2} - \overline{x_1}}{\overline{x_1} - \overline{x_2}} \otimes \overline{x_2} - c \cdot \frac{\overline{x_2} - \overline{x_2}}{\overline{x_1} - \overline{x_3}} \otimes \overline{x_3} + c \cdot \frac{\overline{x_2} - \overline{x_3}}{\overline{x_2} - \overline{x_3}} \otimes \overline{x_1} \\
&= -t \cdot 1 \otimes \overline{x_1} + 2c \cdot 1 \otimes \overline{x_2} + c \cdot 1 \otimes \overline{x_1} \\
&= (-t + c) \cdot 1 \otimes \overline{x_1} + 2c \cdot 1 \otimes \overline{x_2}.
\end{aligned}
$$

$$
\begin{aligned}
D_{y_1-y_2}(\overline{x_1} \otimes \overline{x_2}) &= t\partial_{y_1-y_2}(\overline{x_1}) \otimes \overline{x_2} - c\,\langle y_1 - y_2, \overline{x_1} - \overline{x_2}\rangle \frac{\overline{x_1} - (12).\overline{x_1}}{\overline{x_1} - \overline{x_2}} \otimes (12).\overline{x_2} \\
&\quad - c\,\langle y_1 - y_2, \overline{x_1} - \overline{x_3}\rangle \frac{\overline{x_1} - (13).\overline{x_1}}{\overline{x_1} - \overline{x_3}} \otimes (13).\overline{x_2} \\
&\quad - c\,\langle y_1 - y_2, \overline{x_2} - \overline{x_3}\rangle \frac{\overline{x_1} - (23).\overline{x_1}}{\overline{x_2} - \overline{x_3}} \otimes (23).\overline{x_2} \\
&= t \cdot 1 \otimes \overline{x_2} - 2c \cdot \frac{\overline{x_1} - \overline{x_2}}{\overline{x_1} - \overline{x_2}} \otimes \overline{x_1} - c \cdot \frac{\overline{x_1} - \overline{x_3}}{\overline{x_1} - \overline{x_3}} \otimes \overline{x_2} + c \cdot \frac{\overline{x_1} - \overline{x_1}}{\overline{x_2} - \overline{x_3}} \otimes \overline{x_3} \\
&= t \cdot 1 \otimes \overline{x_2} - 2c \cdot 1 \otimes \overline{x_1} - c \cdot 1 \otimes \overline{x_2} \\
&= -2c \cdot 1 \otimes \overline{x_1} + (t - c) \cdot 1 \otimes \overline{x_2}.
\end{aligned}
$$

$$
\begin{aligned}
D_{y_1-y_2}(\overline{x_2} \otimes \overline{x_2}) &= t\partial_{y_1-y_2}(\overline{x_2}) \otimes \overline{x_2} - c\,\langle y_1 - y_2, \overline{x_1} - \overline{x_2}\rangle \frac{\overline{x_2} - (12).\overline{x_2}}{\overline{x_1} - \overline{x_2}} \otimes (12).\overline{x_2} \\
&\quad - c\,\langle y_1 - y_2, \overline{x_1} - \overline{x_3}\rangle \frac{\overline{x_2} - (13).\overline{x_2}}{\overline{x_1} - \overline{x_3}} \otimes (13).\overline{x_2} \\
&\quad - c\,\langle y_1 - y_2, \overline{x_2} - \overline{x_3}\rangle \frac{\overline{x_2} - (23).\overline{x_2}}{\overline{x_2} - \overline{x_3}} \otimes (23).\overline{x_2} \\
&= -t \cdot 1 \otimes \overline{x_2} - 2c \cdot \frac{\overline{x_2} - \overline{x_1}}{\overline{x_1} - \overline{x_2}} \otimes \overline{x_1} - c \cdot \frac{\overline{x_2} - \overline{x_2}}{\overline{x_1} - \overline{x_3}} \otimes \overline{x_2} + c \cdot \frac{\overline{x_2} - \overline{x_3}}{\overline{x_2} - \overline{x_3}} \otimes \overline{x_3} \\
&= -t \cdot 1 \otimes \overline{x_2} + 2c \cdot 1 \otimes \overline{x_1} + c \cdot 1 \otimes (-\overline{x_1} - \overline{x_2})
\end{aligned}
$$

$$= c \cdot 1 \otimes \overline{x_1} + (-t-c) \cdot 1 \otimes \overline{x_2}.$$

$$\begin{aligned}
D_{y_2-y_3}(\overline{x_1} \otimes \overline{x_1}) &= (12)(23)D_{y_1-y_2}(23)(12)(\overline{x_1} \otimes \overline{x_1}) \\
&= (12)(23)D_{y_1-y_2}(\overline{x_3} \otimes \overline{x_3}) \\
&= (12)(23)D_{y_1-y_2}((-\overline{x_1} - \overline{x_2}) \otimes (-\overline{x_1} - \overline{x_2})) \\
&= (12)(23)D_{y_1-y_2}\left(\overline{x_1} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2}\right) \\
&= (12)(23)\left((t+c-2c-t+c+c) \cdot 1 \otimes \overline{x_1} + (-c+t-c+2c-t-c) \cdot 1 \otimes \overline{x_2}\right) \\
&= (12)(23)\left(c \cdot 1 \otimes \overline{x_1} - c \cdot 1 \otimes \overline{x_2}\right) \\
&= c \cdot 1 \otimes \overline{x_2} - c \cdot 1 \otimes \overline{x_3} \\
&= c \cdot 1 \otimes \overline{x_2} + c \cdot 1 \otimes \overline{x_1} + c \cdot 1 \otimes \overline{x_2} \\
&= c \cdot 1 \otimes \overline{x_1} + 2c \cdot 1 \otimes \overline{x_2}.
\end{aligned}$$

$$\begin{aligned}
D_{y_2-y_3}(\overline{x_2} \otimes \overline{x_1}) &= D_{y_2-y_3}(12)(\overline{x_1} \otimes \overline{x_2}) \\
&= (12)D_{y_1-y_3}(\overline{x_1} \otimes \overline{x_2}) \\
&= (12)\left(D_{y_1-y_2} + D_{y_2-y_3}\right)(\overline{x_1} \otimes \overline{x_2}) \\
&= (12)\left((-2c+c) \cdot 1 \otimes \overline{x_1} + (t-c-c) \cdot 1 \otimes \overline{x_2}\right) \\
&= (t-2c) \cdot 1 \otimes \overline{x_1} + (-c) \cdot 1 \otimes \overline{x_2}.
\end{aligned}$$

$$\begin{aligned}
D_{y_2-y_3}(\overline{x_1} \otimes \overline{x_2}) &= (12)(23)D_{y_1-y_2}(23)(12)(\overline{x_1} \otimes \overline{x_2}) \\
&= (12)(23)D_{y_1-y_2}(\overline{x_3} \otimes \overline{x_1}) \\
&= (12)(23)D_{y_1-y_2}((-\overline{x_1} - \overline{x_2}) \otimes \overline{x_1}) \\
&= (12)(23)D_{y_1-y_2}\left(-\overline{x_1} \otimes \overline{x_1} - \overline{x_2} \otimes \overline{x_1}\right) \\
&= (12)(23)\left(-(t+c-t+c) \cdot 1 \otimes \overline{x_1} - (-c+2c) \cdot 1 \otimes \overline{x_2}\right) \\
&= (12)(23)\left((-2c) \cdot 1 \otimes \overline{x_1} + (-c) \cdot 1 \otimes \overline{x_2}\right) \\
&= (-2c) \cdot 1 \otimes \overline{x_2} + (-c) \cdot 1 \otimes \overline{x_3} \\
&= (-2c) \cdot 1 \otimes \overline{x_2} + c \cdot 1 \otimes \overline{x_1} + c \cdot 1 \otimes \overline{x_2} \\
&= c \cdot 1 \otimes \overline{x_1} + (-c) \cdot 1 \otimes \overline{x_2}.
\end{aligned}$$

$$\begin{aligned}
D_{y_2-y_3}(\overline{x_2} \otimes \overline{x_2}) &= D_{y_2-y_3}(12)(\overline{x_1} \otimes \overline{x_1}) \\
&= (12)D_{y_1-y_3}(\overline{x_1} \otimes \overline{x_1}) \\
&= (12)\left(D_{y_1-y_2} + D_{y_2-y_3}\right)(\overline{x_1} \otimes \overline{x_1}) \\
&= (12)\left((t+c+c) \cdot 1 \otimes \overline{x_1} + (-c+2c) \cdot 1 \otimes \overline{x_2}\right)
\end{aligned}$$

$$= c \cdot 1 \otimes \overline{x_1} + (t + 2c) \cdot 1 \otimes \overline{x_2}.$$

□

Lemma 8.3.2. *1. For any prime p and for $t = 0$, the vectors*

$$v_1 = -\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2} + 2\overline{x_2} \otimes \overline{x_2}$$
$$v_2 = 2\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2} - \overline{x_2} \otimes \overline{x_2}$$

in $M^1_{0,c}(\texttt{stand})$ are in the intersection of the kernels of $D_{y_1-y_2}$ and $D_{y_2-y_3}$.

2. For $p \neq 3$, $t = 0$ and $c \neq 0$, the intersection of the kernels of $D_{y_1-y_2}$ and $D_{y_2-y_3}$ on $M^1_{0,c}(\texttt{stand})$ is $J^1_{0,c}(\texttt{stand})$ which spanned by the vectors v_1 and v_2.

Proof. 1. This follows immediately from Lemma 8.3.1 by setting $t = 0$.

2. The leftmost 2×2 minor of the matrix of $(D_{y_1-y_2})|_{M^1_{t,c}(\texttt{stand})}$ has the determinant

$$(t + c)(2c) - (-c)(-t + c) = c(t + 3c)$$

For $p \neq 3$, $t = 0$ and $c \neq 0$ this determinant is nonzero, showing that $(D_{y_1-y_2})|_{M^1_{t,c}(\texttt{stand})}$ has rank 2 and thus its kernel has dimension at most 2.

□

Lemma 8.3.3. *Let $p \neq 3$, $t = 0$ and $c \neq 0$.*

1. *The vectors $v_1, v_2 \in M^1_{0,c}(\texttt{stand})$ from Lemma 8.3.2 generate a subrepresentation of $M_{0,c}(\texttt{stand})$ isomorphic to $M_{0,c}(\texttt{stand})[-1]$.*

2. *The Hilbert series of the quotient of the Verma module $M_{0,c}(\texttt{stand})$ by the submodule generated by v_1, v_2 is*
$$\frac{2(1 - z)}{(1 - z)^2}.$$

3. *The singular vectors $\overline{\sigma_2} \otimes \overline{x_1}$ and $\overline{\sigma_2} \otimes \overline{x_2}$ are in the submodule generated by v_1, v_2.*

4. *The singular vectors $\overline{\sigma_3} \otimes \overline{x_1}$ and $\overline{\sigma_3} \otimes \overline{x_2}$ are not in the submodule generated by v_1, v_2.*

5. *The quotient of the Verma module $M_{0,c}(\texttt{stand})$ by the submodule generated by v_1, v_2, $\overline{\sigma_3} \otimes \overline{x_1}, \overline{\sigma_3} \otimes \overline{x_2}$ has the character*

$$[\texttt{stand}] + ([\texttt{triv}] + [\texttt{sign}])z + [\texttt{stand}]z^2$$

and the Hilbert polynomial
$$2 + 2z + 2z^2.$$

6. *The quotient of the Verma module $M_{0,c}(\texttt{stand})$ by the submodule generated by v_1, v_2, $\overline{\sigma_3} \otimes \overline{x_1}, \overline{\sigma_3} \otimes \overline{x_2}$ is irreducible.*

Proof. 1. The vectors v_1, v_2 are singular and span a representation isomorphic to the standard representation with the isomorphism

$$\varphi : \texttt{stand} \to \operatorname{span}\{v_1, v_2\}$$

given by

$$\varphi(\overline{x_i}) = v_i.$$

By the universal mapping property of the induced module $M_{0,c}(\texttt{stand})$, the homomorphism φ of S_3 representations extends to a homomorphism, also denoted φ, of graded $H_{0,c}(S_3, \mathfrak{h})$ representations

$$\varphi : M_{0,c}(\texttt{stand})[-1] \to M_{0,c}(\texttt{stand}).$$

where the notation $M[k]$ denotes the grading shift by k meaning $M^i[k] = M^{i+k}$. For $f \in S(\mathfrak{h}^*)$ and $i = 1, 2$ we have

$$\varphi(f \otimes \overline{x_i}) = f \cdot v_i.$$

We claim φ is injective.

Let $v \in M_{0,c}(\texttt{stand})$ be some vector in the kernel of φ. Without loss of generality assume v is homogeneous, so there are $A, B \in S(\mathfrak{h}^*)$ homogeneous of the same degree such that

$$v = A \otimes \overline{x_1} + B \otimes \overline{x_2}.$$

Applying φ we get

$$Av_1 + Bv_2 = 0,$$

which becomes

$$\begin{aligned} &A\left(-\overline{x_1} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2} + \overline{x_2} \otimes \overline{x_1} + 2\overline{x_2} \otimes \overline{x_2}\right) \\ &+B\left(2\overline{x_1} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2} + \overline{x_2} \otimes \overline{x_1} - \overline{x_2} \otimes \overline{x_2}\right) = 0. \end{aligned}$$

Reading off the coefficients of $\otimes\overline{x_1}$ and $\otimes\overline{x_2}$, we get that this is equivalent to

$$\begin{aligned} (-\overline{x_1} + \overline{x_2})A + (2\overline{x_1} + \overline{x_2})B &= 0, \\ (\overline{x_1} + 2\overline{x_2})A + (\overline{x_1} - \overline{x_2})B &= 0. \end{aligned}$$

Considering this as a system of equations over the rational function field $\Bbbk(\mathfrak{h}^*)$ with unknowns $A, B \in S(\mathfrak{h}^*)$, we calculate its determinant

$$\begin{aligned}\Delta &= (-\overline{x_1} + \overline{x_2})(\overline{x_1} - \overline{x_2}) - (2\overline{x_1} + \overline{x_2})(\overline{x_1} + 2\overline{x_2}) \\ &= -3(\overline{x_1}^2 + \overline{x_1 x_2} + \overline{x_2}^2) \\ &= -3\overline{\sigma_2}.\end{aligned}$$

So, whenever $p \neq 3$, the above system has only the trivial solution $A = B = 0$, so the kernel of the $H_{0,c}(S_3, \mathfrak{h})$ homomorphism φ contains only $v = 0$, and φ is injective. Its image is the subrepresentation of $M_{0,c}(\texttt{stand})$ generated by v_1, v_2, and φ gives an isomorphism between $M_{0,c}(\texttt{stand})[-1]$ and this subrepresentation of $M_{0,c}(\texttt{stand})$.

2. Consequently, the Hilbert series of the subrepresentation generated by v_1, v_2 is

$$z \cdot \mathrm{Hilb}_{M_{0,c}(\texttt{stand})}(z) = \frac{2z}{(1-z)^2}$$

and the Hilbert series of the quotient of the Verma module $M_{0,c}(\texttt{stand})$ by the submodule generated by v_1, v_2 is

$$\frac{2}{(1-z)^2} - \frac{2z}{(1-z)^2} = \frac{2(1-z)}{(1-z)^2}.$$

3. Checking whether a given vector $u = u_1 \otimes \overline{x_1} + u_2 \otimes \overline{x_2}$ is in the subrepresentation generated by v_1, v_2 is equivalent to solving the system

$$Av_1 + Bv_2 = u,$$

for $A, B \in S(\mathfrak{h}^*)$. Reading off the coefficients of $\otimes\overline{x_1}$ and $\otimes\overline{x_2}$ we get an equivalent to the system of equations

$$\begin{aligned}(-\overline{x_1} + \overline{x_2})A + (2\overline{x_1} + \overline{x_2})B &= u_1, \\ (\overline{x_1} + 2\overline{x_2})A + (\overline{x_1} - \overline{x_2})B &= u_2.\end{aligned}$$

This system has a determinant $\Delta = -3\overline{\sigma_2}$. When $p \neq 3$, its unique solutions A, B, as rational functions on $\mathfrak{h}$, as given as:

$$\begin{aligned}A &= \frac{u_1(\overline{x_1} - \overline{x_2}) + u_2(2\overline{x_1} + \overline{x_2})}{-3\overline{\sigma_2}}, \\ B &= \frac{u_1(\overline{x_1} + 2\overline{x_2}) + u_2(\overline{x_1} - \overline{x_2})}{3\overline{\sigma_2}}.\end{aligned} \tag{8.3.4}$$

The question of whether $u = u_1 \otimes \overline{x_1} + u_2 \otimes \overline{x_2}$ is in the subrepresentation generated by v_1, v_2 is equivalent to checking whether these A, B are polynomials in $\mathfrak{h}^*$ (as opposed to rational functions).

When $u = \overline{\sigma_2} \otimes \overline{x_1}$ we have $u_1 = \overline{\sigma_2}$, $u_2 = 0$ so the solutions to (8.3.4) are indeed polynomial, with

$$A = \frac{-1}{3}(\overline{x_1} - \overline{x_2}), \qquad B = \frac{1}{3}(\overline{x_1} + 2\overline{x_2}).$$

When $u = \overline{\sigma_2} \otimes \overline{x_2}$ we have $u_1 = 0$, $u_2 = \overline{\sigma_2}$, and the solutions to (8.3.4) are indeed polynomial, with

$$A = \frac{-1}{3}(2\overline{x_1} + \overline{x_2}), \qquad B = \frac{1}{3}(\overline{x_1} - \overline{x_2}).$$

This shows that $\overline{\sigma_2} \otimes \overline{x_1}$ and $\overline{\sigma_2} \otimes \overline{x_2}$ are in the subrepresentation generated by v_1, v_2.

4. Similarly, considering whether $\overline{\sigma_3} \otimes \overline{x_1}$ or $\overline{\sigma_3} \otimes \overline{x_2}$ are in the subrepresentation generated by v_1, v_2 is equivalent to checking whether substituting $u_1 = \overline{\sigma_3}, u_2 = 0$ and $u_1 = 0, u_2 = \overline{\sigma_3}$ into (8.3.4) gives polynomial solutions A, B.

   When $u_1 = \overline{\sigma_3}, u_2 = 0$ the formulas for the solution to the system (8.3.4) are

$$A = \frac{\overline{\sigma_3}(\overline{x_1} - \overline{x_2})}{-3\overline{\sigma_2}} = \frac{\overline{x_1}\overline{x_2}(\overline{x_1} + \overline{x_2})(\overline{x_1} - \overline{x_2})}{3(\overline{x_1}^2 + \overline{x_1}\overline{x_2} + \overline{x_2}^2)},$$

$$B = \frac{\overline{\sigma_3}(\overline{x_1} + 2\overline{x_2})}{3\overline{\sigma_2}} = \frac{\overline{x_1}\overline{x_2}(\overline{x_1} + \overline{x_2})(\overline{x_1} + 2\overline{x_2})}{-3(\overline{x_1}^2 + \overline{x_1}\overline{x_2} + \overline{x_2}^2)}.$$

   In characteristics other than $p = 3$ the numerators and denominators of these rational functions are coprime, so A, B are not polynomials. We conclude that $\overline{\sigma_3} \otimes \overline{x_1}$ is not in the subrepresentation generated by v_1, v_2.

   Similar argument gives that $\overline{\sigma_3} \otimes \overline{x_2}$ is not in the subrepresentation generated by v_1, v_2.

5. Let $M_{0,c}(\texttt{stand})/U$ be the quotient of the Verma module $M_{0,c}(\texttt{stand})$ by the submodule U generated by all the singular vectors found so far - v_1, v_2 in degree 1, $\overline{\sigma_2} \otimes \overline{x_1}, \overline{\sigma_2} \otimes \overline{x_2}$ in degree 2 and $\overline{\sigma_3} \otimes \overline{x_1}, \overline{\sigma_3} \otimes \overline{x_2}$ in degree 3. By the argument in part (3) of this Lemma, U is also generated by v_1, v_2 in degree 1 and $\overline{\sigma_3} \otimes \overline{x_1}, \overline{\sigma_3} \otimes \overline{x_2}$ in degree 3.

   Degree 0 of the quotient $M_{0,c}(\texttt{stand})/U$ is equal to $M^0_{0,c}(\texttt{stand})$, so it is isomorphic to `stand` as an S_3 representation. Degree 1 of the quotient $M_{0,c}(\texttt{stand})/U$ is isomorphic to $M^1_{0,c}(\texttt{stand})/\mathrm{span}\{v_1, v_2\}$, so in the Grothendieck group is equal to

$$[S^1(\mathfrak{h}^*) \otimes \texttt{stand}] - [\texttt{stand}] = [\texttt{stand}] \cdot [\texttt{stand}] - [\texttt{stand}] = [\texttt{triv}] + [\texttt{sign}].$$

   Similarly, using part (1) of this Lemma, degree 2 of the quotient $M_{0,c}(\texttt{stand})/U$ is, in the Grothendieck group, equal to

$$[S^2(\mathfrak{h}^*) \otimes \texttt{stand}] - [S^1(\mathfrak{h}^*) \otimes \texttt{stand}]$$
$$= (2 \cdot [\texttt{stand}] + [\texttt{triv}] + [\texttt{sign}]) - ([\texttt{stand}] + [\texttt{triv}] + [\texttt{sign}]) = [\texttt{stand}].$$

Finally, using part (1) of this Lemma, degree 3 of the quotient of $M_{0,c}(\texttt{stand})$ by the submodule generated by v_1, v_2 looks like

$$\begin{aligned}&[S^3(\mathfrak{h}^*)\otimes \texttt{stand}] - [S^2(\mathfrak{h}^*)\otimes \texttt{stand}]\\ &\qquad = (3\cdot[\texttt{stand}]+[\texttt{triv}]+[\texttt{sign}]) - (2[\texttt{stand}]+[\texttt{triv}]+[\texttt{sign}]) = [\texttt{stand}],\end{aligned}$$

so, using part (4) of this Lemma, the quotient by the further singular vectors in degree 3, namely $\overline{\sigma_3}\otimes\overline{x_1}, \overline{\sigma_3}\otimes\overline{x_2}$ which span an S_3 subrepresentation isomorphic to stand gives the character of $[\texttt{stand}]-[\texttt{stand}]=0$ in degree 3.

All together, this means that for $p\neq 3$ the character of $M_{0,c}(\texttt{stand})/U$ is

$$[\texttt{stand}] + ([\texttt{triv}]+[\texttt{sign}])z + [\texttt{stand}]z^2$$

and the Hilbert polynomial is

$$2+2z+2z^2.$$

6. When $p>3$, [DeSa14] Proposition 4.1 tells us that the Hilbert polynomial of the irreducible representation $L_{0,c}(\texttt{stand})$ is $2+2z+2z^2$. As $L_{0,c}(\texttt{stand})$ is the quotient by a maximal submodule of $M_{0,c}(\texttt{stand})$, and $M_{0,c}(\texttt{stand})/U$ is a quotient, it follows that $L_{0,c}(\texttt{stand})$ is a quotient of $M_{0,c}(\texttt{stand})/U$. Using that $M_{0,c}(\texttt{stand})/U$ and $L_{0,c}(\texttt{stand})$ have the same Hilbert polynomial, it follows they are equal.

   When $p=2$ the results of [DeSa14] do not apply to $H_{t,c}(S_3,\mathfrak{h})$. We proceed by a direct computation which works for all $p\neq 3$.

   If the module $M_{0,c}(\texttt{stand})/U$ is not irreducible, then it has singular vectors. By definition those cannot be in degree 0, and we showed in Lemma 8.3.2 part (2) that there are no singular vectors in degree 1 of $M_{0,c}(\texttt{stand})/U$. It remains to examine degree 2.

   Degree 2 of $M_{0,c}(\texttt{stand})$ has a basis

$$\{\overline{x_1}^2\otimes\overline{x_1},\ \overline{x_1x_2}\otimes\overline{x_1},\ \overline{x_2}^2\otimes\overline{x_1},\ \overline{x_1}^2\otimes\overline{x_2},\ \overline{x_1x_2}\otimes\overline{x_2},\ \overline{x_2}^2\otimes\overline{x_2}\}.$$

   Taking the quotient by the submodule generated by v_1, v_2 has the effect on degree 2 of $M_{0,c}(\texttt{stand})/U$ of taking the quotient by the vector space spanned by $x_1v_1, x_2v_1, x_1v_2, x_2v_2$. Degree 2 of the quotient $M_{0,c}(\texttt{stand})/U$ is spanned by the images of

$$\overline{x_1}^2\otimes\overline{x_1}, \overline{x_1}^2\otimes\overline{x_2}.$$

   Let us calculate the action of the Dunkl operators on them:

$$\begin{aligned}D_{y_1-y_2}(\overline{x_1}^2\otimes\overline{x_1}) &= -c\left(2(\overline{x_1}+\overline{x_2})\otimes\overline{x_2} + (\overline{x_1}+\overline{x_3})\otimes\overline{x_3}\right)\\ &= -c\left(2\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1}+3\overline{x_2}\otimes\overline{x_2}\right),\end{aligned}$$

$$\begin{aligned} D_{y_1-y_2}(\overline{x_1}^2 \otimes \overline{x_2}) &= -c\,(2(\overline{x_1}+\overline{x_2}) \otimes \overline{x_1} + (\overline{x_1}+\overline{x_3}) \otimes \overline{x_2}) \\ &= -c\,(2\overline{x_1} \otimes \overline{x_1} + 2\overline{x_2} \otimes \overline{x_1} - \overline{x_2} \otimes \overline{x_2})\,. \end{aligned}$$

In the 4-dimensional space $M^1_{0,c}(\texttt{stand})$, the vectors v_1, v_2, $D_{y_1-y_2}(\overline{x_1}^2 \otimes \overline{x_1})$ and $D_{y_1-y_2}(\overline{x_1}^2 \otimes \overline{x_2})$, are linearly independent. This means that in the quotient

$$\big(M_{0,c}(\texttt{stand})/U\big)^1 = \big(M_{0,c}(\texttt{stand})/\mathrm{span}\{v_1, v_2\}\big)^1,$$

no nontrivial linear combination of $D_{y_1-y_2}(\overline{x_1}^2 \otimes \overline{x_1})$ and $D_{y_1-y_2}(\overline{x_1}^2 \otimes \overline{x_2})$ is zero. This further means that no linear combination of $\overline{x_1}^2 \otimes \overline{x_1}$ and $\overline{x_1}^2 \otimes \overline{x_2}$ is singular in $M_{0,c}(\texttt{stand})/U$.

We conclude that $M_{0,c}(\texttt{stand})/U$ is irreducible.

□

Summarising the results of the last lemma, we get:

Corollary 8.3.5. *For any $p \neq 3$ and $c \neq 0$ the irreducible representation $L_{0,c}(\texttt{stand})$ is the quotient of the Verma module $M_{0,c}(\texttt{stand})$ by the submodule generated by vectors v_1, v_2 from Lemma 8.3.2 and $\overline{\sigma_3} \otimes \overline{x_1}$, $\overline{\sigma_3} \otimes \overline{x_2}$. Its character and Hilbert polynomial are:*

$$\chi_{L_{0,c}(\texttt{stand})}(z) = \begin{cases} [\texttt{stand}] + 2[\texttt{triv}]z + [\texttt{stand}]z^2, & p = 2 \\ [\texttt{stand}] + ([\texttt{triv}] + [\texttt{sign}])z + [\texttt{stand}]z^2, & p > 3 \end{cases}$$

$$Hilb_{L_{0,c}(\texttt{stand})}(z) = 2 + 2z + 2z^2.$$

Proof. By Lemma 8.3.3 part (6), $L_{0,c}(\texttt{stand})$ is the quotient of the Verma module $M_{0,c}(\texttt{stand})$ by the submodule generated by $v_1, v_2, \overline{\sigma_3} \otimes \overline{x_1}, \overline{\sigma_3} \otimes \overline{x_2}$. By Lemma 8.3.3 part (5), this module has the required character and Hilbert polynomial. It remains only to observe that for $p = 2$, $[\texttt{triv}] = [\texttt{sign}]$. □

8.4 The irreducible representation $L_{1,c}(\texttt{stand})$ in characteristic 2 for generic c

For this entire section, let $p = 2$, $t = 1$, and let c be generic. We will describe the irreducible representation $L_{1,c}(\texttt{stand})$, give singular vectors and its character and Hilbert polynomial, and clarify again that "c generic" here means $c \notin \mathbb{F}_2$.

The strategy we use is similar to the strategy we use later for $L_{1,c}(\texttt{stand})$ in characteristic $p > 3$, but some of the computations we use there do not work in characteristic 2 (namely, the rescaled Young basis $\{b_+, b_-\}$ we use there is not a basis in characteristic 2, as explained in Section 7.2) so we need to treat this case separately.

First we check what the action of the Casimir element Ω (Definition 2.3.1) tells us about this case.

Lemma 8.4.1. *Let f be a homogeneous vector in a representation of $H_{1,c}(S_3,\mathfrak{h})$ over a field of characteristic 2, and an element of an irreducible S_3 subrepresentation τ. If f is singular, then*

$$\Omega . f = \begin{cases} 0 & \tau = \mathtt{triv}, \\ cf & \tau = \mathtt{stand}. \end{cases}$$

Proof. As $y.f = 0$ for all $y \in \mathfrak{h}$,

$$\Omega . f = \sum_{s \in S} c(1-s).f = \begin{cases} 0 & \tau = \mathtt{triv}, \\ 3cf & \tau = \mathtt{stand}, \end{cases}$$
$$= \begin{cases} 0 & \tau = \mathtt{triv}, \\ cf & \tau = \mathtt{stand}. \end{cases}$$

□

Lemma 8.4.2. *The following vectors in $M^2_{1,c}(\mathtt{stand})$ span an S_3 subrepresentation isomorphic to* stand *and are singular for every c over fields of characteristic $p = 2$:*

$$v_1 = c\overline{\sigma_2} \otimes \overline{x_1} + (\overline{x_1}^2 \otimes \overline{x_1} + \overline{x_1}^2 \otimes \overline{x_2} + \overline{x_2}^2 \otimes \overline{x_1})$$
$$v_2 = c\overline{\sigma_2} \otimes \overline{x_2} + (\overline{x_2}^2 \otimes \overline{x_2} + \overline{x_1}^2 \otimes \overline{x_2} + \overline{x_2}^2 \otimes \overline{x_1})$$

Proof. By Theorem 7.2.9, v_1, v_2 span a copy of stand, with the isomorphism given by $\overline{x_i} \mapsto v_i$. Let us first show that

$$\begin{aligned} v_1 + v_2 &= c\overline{\sigma_2} \otimes (\overline{x_1} + \overline{x_2}) + \overline{x_1}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_2} \\ &= (c+1)\overline{x_1}^2 \otimes \overline{x_1} + c\overline{x_1 x_2} \otimes \overline{x_1} + c\overline{x_2}^2 \otimes \overline{x_1} + c\overline{x_1}^2 \otimes \overline{x_2} + c\overline{x_1 x_2} \otimes \overline{x_2} + (c+1)\overline{x_2}^2 \otimes \overline{x_2} \end{aligned}$$

is singular.

$$\begin{aligned} D_{y_1 - y_3}(v_1 + v_2) &= \big(\partial_{y_1 - y_3} \otimes \mathrm{id}\big)(v_1 + v_2) - c \sum_{(ij) \in S} \langle \overline{x_i} - \overline{x_j}, y_1 - y_3 \rangle \left(\frac{(1-(ij))}{\overline{x_i} - \overline{x_j}} \otimes (ij) \right)(v_1 + v_2) \\ &= c\overline{x_2} \otimes (\overline{x_1} + \overline{x_2}) - c\big((\overline{x_1} + \overline{x_2}) \otimes \overline{x_2} + (\overline{x_1} + \overline{x_2}) \otimes \overline{x_1} + (\overline{x_2} + \overline{x_3}) \otimes \overline{x_3}\big) \\ &= c\overline{x_2} \otimes (\overline{x_1} + \overline{x_2}) - c\big((\overline{x_1} + \overline{x_2}) \otimes \overline{x_2} + (\overline{x_1} + \overline{x_2}) \otimes \overline{x_1} + \overline{x_1} \otimes (\overline{x_1} + \overline{x_2})\big) \\ &= 0. \end{aligned}$$

From here it follows that

$$\begin{aligned} D_{y_2-y_3}(v_1+v_2) &= (12)D_{y_1-y_3}(12)(v_1+v_2) \\ &= (12)D_{y_1-y_3}(v_1+v_2) = (12).0 = 0. \end{aligned}$$

This shows that $v_1 + v_2$ is singular. From there we can conclude that the all vectors in the irreducible S_3 representation in which $v_1 + v_2$ lies are singular, and in particular so are v_1, v_2.

□

Checking whether some vector $u = u_1 \otimes \overline{x_1} + u_2 \otimes \overline{x_2} \in M_{1,c}(\texttt{stand})$ is in the subrepresentation generated by v_1, v_2 is equivalent to solving the system

$$Av_1 + Bv_2 = u,$$

for $A, B \in S(\mathfrak{h}^*)$. Reading off the coefficients of $\otimes\overline{x_1}$ and $\otimes\overline{x_2}$ we get this is equivalent to the system of equations

$$\begin{aligned} &\big((c+1)\overline{x_1}^2 + c\overline{x_1x_2} + (c+1)\overline{x_2}^2\big)A + \overline{x_2}^2 B = u_1, \\ &\overline{x_1}^2 A + \big((c+1)\overline{x_1}^2 + c\overline{x_1x_2} + (c+1)\overline{x_2}^2\big)B = u_2. \end{aligned} \tag{8.4.3}$$

Lemma 8.4.4. *For $p = 2$, the determinant of the system* (8.4.3) *is*

$$\Delta = (c+1)^2\overline{\sigma_2}^2.$$

When $c \neq 1$, the unique rational functions A, B on $\mathfrak{h}$ solving that system are:

$$\begin{aligned} A &= \frac{u_1((c+1)\overline{x_1}^2 + c\overline{x_1x_2} + (c+1)\overline{x_2}^2) - u_2\overline{x_2}^2}{(c+1)^2\overline{\sigma_2}^2} \\ B &= \frac{-u_1\overline{x_1}^2 + u_2((c+1)\overline{x_1}^2 + c\overline{x_1x_2} + (c+1)\overline{x_2}^2)}{(c+1)^2\overline{\sigma_2}^2}. \end{aligned} \tag{8.4.4}$$

Proof. We calculate directly:

$$\begin{aligned} \Delta &= ((c+1)\overline{x_1}^2 + c\overline{x_1x_2} + (c+1)\overline{x_2}^2)^2 - \overline{x_1}^2\overline{x_2}^2 \\ &= (c+1)^2(\overline{x_1}^2 + \overline{x_1x_2} + \overline{x_2}^2)^2 \\ &= (c+1)^2\overline{\sigma_2}^2. \end{aligned}$$

□

We proceed in a way analogous to Lemma 8.3.3.

Lemma 8.4.5. *When $p = 2$, $t = 1$ and $c \neq 1$, the vectors $v_1, v_2 \in M^2_{1,c}(\texttt{stand})$ from Lemma 8.4.2 generate a subrepresentation of $M_{1,c}(\texttt{stand})$ isomorphic to $M_{1,c}(\texttt{stand})[-2]$. The Hilbert series of the quotient of $M_{1,c}(\texttt{stand})$ by this submodule is $\frac{2(1-z^2)}{(1-z)^2}$.*

Proof. The isomorphism of S_3 representations

$$\varphi : \texttt{stand} \to \operatorname{span}\{v_1, v_2\}$$

given by $\varphi(\overline{x_i}) = v_i$ extends to a homomorphism of graded $H_{1,c}(S_3, \mathfrak{h})$ modules

$$\varphi : M_{1,c}(\texttt{stand})[-2] \to M_{1,c}(\texttt{stand}).$$

Assuming some nonzero vector is in the kernel of φ is equivalent to assuming there is a nontrivial solution $A, B \in S(\mathfrak{h}^*)$ to the equation

$$Av_1 + Bv_2 = 0.$$

This is equivalent to there existing a nontrivial solution to the system (8.4.3) with $u_1 = u_2 = 0$, which is impossible because by Lemma 8.4.4 the determinant of this system is $\Delta = (c+1)^2\overline{\sigma_2}^2 \neq 0$. □

Next, we consider the quotient $M_{1,c}(\texttt{stand})/\langle v_1, v_2\rangle$, and check to see if the p-th powers of the invariants generate proper submodules in this quotient.

Lemma 8.4.6. *Let $p = 2$, $t = 1$, $c \neq 1$. We have*

$$\overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2} \in \langle v_1, v_2\rangle, \qquad \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2} \notin \langle v_1, v_2\rangle.$$

Proof. A vector $\overline{\sigma_j}^2 \otimes \overline{x_i}$, $i = 1, 2$, $j = 2, 3$ is in the submodule generated by v_1, v_2 if and only if the system (8.4.3) with $u_i = \overline{\sigma_j}^2$, $u_k = 0$ for $k \neq i$ has polynomial solutions $A, B \in S(\mathfrak{h}^*)$. Lemma 8.4.4 gives these solutions explicitly as rational functions. If $j = 2$, these rational functions are in fact polynomial, as the factors $\overline{\sigma_j}^2$ in the numerator and the denominator cancel. If $j = 3$, these rational functions are not polynomial, as the numerator and the denominator are coprime. □

Next, we consider the quotient $M_{1,c}(\texttt{stand})/\langle v_1, v_2, \overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2}, \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2}\rangle$. First we use Lemma 8.4.5 to calculate its character, and then we will show this module is in fact irreducible and thus equal to $L_{1,c}(\texttt{stand})$.

Lemma 8.4.7. *Let $p = 2$, $t = 1$, $c \neq 1$. The module*

$$M_{1,c}(\texttt{stand})/\langle v_1, v_2, \overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2}, \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2}\rangle$$

has the character

$$[\texttt{stand}](1+z+z^2+2z^3+z^4+z^5+z^6)+2[\texttt{triv}](z+z^2+z^4+z^5)$$

and the Hilbert polynomial

$$2(1+2z+2z^2+2z^3+2z^4+2z^5+z^6)=2\frac{(1-z^2)(1-z^6)}{(1-z)^2}.$$

Proof. By Theorem 7.2.9, the first terms of the character of the Verma module $M_{1,c}(\texttt{stand})\cong S(\mathfrak{h}^*)\otimes\texttt{stand}$ are

$$\begin{aligned}\chi_{M_{1,c}(\texttt{stand})}(z)=&[\texttt{stand}]+\big([\texttt{stand}]+2[\texttt{triv}]\big)z+\big(2[\texttt{stand}]+2[\texttt{triv}]\big)z^2\\&+\big(3[\texttt{stand}]+2[\texttt{triv}]\big)z^3+\big(3[\texttt{stand}]+4[\texttt{triv}]\big)z^4+\big(4[\texttt{stand}]+4[\texttt{triv}]\big)z^5\\&+\big(5[\texttt{stand}]+4[\texttt{triv}]\big)z^6+\big(5[\texttt{stand}]+6[\texttt{triv}]\big)z^7+\dots\end{aligned}$$

By Lemma 8.4.5, the quotient $M_{1,c}(\texttt{stand})/\left\langle v_1,v_2\right\rangle$ has the character

$$\begin{aligned}\chi_{M_{1,c}(\texttt{stand})}(z)\cdot(1-z^2)=&[\texttt{stand}]+\big([\texttt{stand}]+2[\texttt{triv}]\big)z+\big([\texttt{stand}]+2[\texttt{triv}]\big)z^2\\&+\big(2[\texttt{stand}]\big)z^3+\big([\texttt{stand}]+2[\texttt{triv}]\big)z^4+\big([\texttt{stand}]+2[\texttt{triv}]\big)z^5\\&+\big(2[\texttt{stand}]\big)z^6+\big([\texttt{stand}]+2[\texttt{triv}]\big)z^7+\dots\end{aligned}$$

By Lemma 8.4.6, the vectors $\overline{\sigma_2}^2\otimes\overline{x_1}$ and $\overline{\sigma_2}^2\otimes\overline{x_2}$ are in the submodule of $M_{1,c}(\texttt{stand})$ generated by v_1,v_2, so taking the further quotient by the submodule generated by $\overline{\sigma_2}^2\otimes\overline{x_1}$ and $\overline{\sigma_2}^2\otimes\overline{x_2}$ does not change the character. Also by Lemma 8.4.6, the vectors $\overline{\sigma_3}^2\otimes\overline{x_1}$ and $\overline{\sigma_3}^2\otimes\overline{x_2}$ are not in the submodule of $M_{1,c}(\texttt{stand})$ generated by v_1,v_2. The vectors $\overline{\sigma_3}^2\otimes\overline{x_1}$ and $\overline{\sigma_3}^2\otimes\overline{x_2}$ generate a submodule of $M_{1,c}(\texttt{stand})$ isomorphic to $M_{1,c}(\texttt{stand})$ with a grading shift. To work out what submodule they generate in $M_{1,c}(\texttt{stand})/\left\langle v_1,v_2\right\rangle$, we need to get some information on the first few terms of the character of the intersection $\left\langle v_1,v_2\right\rangle\cap\left\langle\overline{\sigma_3}^2\otimes\overline{x_1},\overline{\sigma_3}^2\otimes\overline{x_2}\right\rangle$.

Assume the intersection $\left\langle v_1,v_2\right\rangle\cap\left\langle\overline{\sigma_3}^2\otimes\overline{x_1},\overline{\sigma_3}^2\otimes\overline{x_2}\right\rangle$ is nonzero in degree 7. This intersection is a submodule of $\left\langle\overline{\sigma_3}^2\otimes\overline{x_1},\overline{\sigma_3}^2\otimes\overline{x_2}\right\rangle$, which is itself a homomomorphic image of $M_{1,c}(\texttt{stand})[-6]$, so isomorphic to a quotient of $M_{1,c}(\texttt{stand})[-6]$. If the intersection were nonzero in degree 7, then $M_{1,c}(\texttt{stand})[-6]$ would have a nontrivial submodule starting in degree 7, and thus $M_{1,c}(\texttt{stand})$ would have a nontrivial submodule starting in degree 1. This means there would be a singular vector in degree 1 of $M_{1,c}(\texttt{stand})$. By Lemma 8.4.1, the action of Ω on $M^1_{1,c}(\texttt{stand})$ is by $c+1$, its action on singular vectors of type `stand` is by c, and its action on singular vectors of type `triv` is by 0. As $c+1\neq c$ and $c+1\neq 0$ whenever $c\neq 1$, $M^1_{1,c}(\texttt{stand})$ doesn't have any singular vectors and so the intersection $\left\langle v_1,v_2\right\rangle\cap\left\langle\overline{\sigma_3}^2\otimes\overline{x_1},\overline{\sigma_3}^2\otimes\overline{x_2}\right\rangle$ is zero in degree 7.

Thus, we can calculate that the first seven terms of the character of the quotient of the Verma module $M_{1,c}(\texttt{stand})$ by the submodule generated by $v_1, v_2, \overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2}, \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2}$ are equal to

$$\begin{aligned}&[\texttt{stand}] + \big([\texttt{stand}] + 2[\texttt{triv}]\big)z + \big([\texttt{stand}] + 2[\texttt{triv}]\big)z^2 \\ &+ \big(2[\texttt{stand}]\big)z^3 + \big([\texttt{stand}] + 2[\texttt{triv}]\big)z^4 + \big([\texttt{stand}] + 2[\texttt{triv}]\big)z^5 + \\ &+ \big([\texttt{stand}]\big)z^6 + 0 \cdot z^7 + \dots\end{aligned}$$

This means the module $M_{1,c}(\texttt{stand})/\langle v_1, v_2, \overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2}, \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2}\rangle$ is zero in degree 7, and consequently in all higher degrees. So, the first terms of its character, which we have calculated, are in fact all, and its character is equal to

$$[\texttt{stand}](1 + z + z^2 + 2z^3 + z^4 + z^5 + z^6) + 2[\texttt{triv}](z + z^2 + z^4 + z^5)$$

as claimed. The Hilbert polynomial follows from taking graded dimensions of both sides. □

Alternative proof. We include this more direct proof, which is similar to a later proof used in the case of $p > 3$.

By Lemma 8.4.5, the quotient $M_{1,c}(\texttt{stand})/\langle v_1, v_2\rangle$ has the character

$$\chi_{M_{1,c}(\texttt{stand})}(z) \cdot (1 - z^2).$$

By Lemma 8.4.6,

$$\langle v_1, v_2\rangle = \left\langle v_1, v_2, \overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2}\right\rangle$$

and

$$\langle v_1, v_2\rangle \neq \left\langle v_1, v_2, \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2}\right\rangle.$$

So

$$\begin{aligned}&\chi_{M_{1,c}(\texttt{stand})/\left\langle v_1, v_2, \overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2}, \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2}\right\rangle} \\ &= \chi_{M_{1,c}(\texttt{stand})} - \chi_{\langle v_1, v_2\rangle} - \chi_{\left\langle \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2}\right\rangle} + \chi_{\langle v_1, v_2\rangle \cap \left\langle \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2}\right\rangle} \\ &= \chi_{M_{1,c}(\texttt{stand})} - z^2 \chi_{M_{1,c}(\texttt{stand})} - z^6 \chi_{M_{1,c}(\texttt{stand})} + \chi_{\langle v_1, v_2\rangle \cap \left\langle \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2}\right\rangle}.\end{aligned}$$

The remaining task is to describe the submodule $\langle v_1, v_2\rangle \cap \left\langle \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2}\right\rangle$. Assume some vector u is in $\langle v_1, v_2\rangle \cap \left\langle \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2}\right\rangle$. This means that there are $A, B, C, D \in S(\mathfrak{h}^*)$, which we can assume without loss of generality are homogeneous, such that

$$u = Av_1 + Bv_2 = C\overline{\sigma_3}^2 \otimes \overline{x_1} + D\overline{\sigma_3}^2 \otimes \overline{x_2}.$$

Solving the equation for C, D we get

$$C = \frac{1}{\overline{\sigma_3}^2} \left(A \cdot ((c+1)\overline{x_1}^2 + c\overline{x_1 x_2} + (c+1)\overline{x_2}^2) + B \cdot \overline{x_2}^2 \right),$$
$$D = \frac{1}{\overline{\sigma_3}^2} \left(A \cdot \overline{x_1}^2 + B \cdot ((c+1)\overline{x_1}^2 + c\overline{x_1 x_2} + (c+1)\overline{x_2}^2) \right).$$

So, we are looking for $A, B \in S(\mathfrak{h}^*)$ such that

$$\overline{\sigma_3}^2 \mid A \cdot (c\overline{\sigma_2} + \overline{x_1}^2 + \overline{x_2}^2) + B \cdot \overline{x_2}^2$$
$$\overline{\sigma_3}^2 \mid A \cdot \overline{x_1}^2 + B \cdot (c\overline{\sigma_2} + \overline{x_1}^2 + \overline{x_2}^2).$$

In particular, taking their linear combination which eliminates B, we get that

$$\overline{\sigma_3}^2 \mid A \cdot (c\overline{\sigma_2} + \overline{x_1}^2 + \overline{x_2}^2)^2 - A \cdot \overline{x_1}^2 \overline{x_2}^2 = A \cdot (c+1)^2 \overline{\sigma_2}^2.$$

As $\overline{\sigma_2}$ and $\overline{\sigma_3}$ have no common factors, it follows that $\overline{\sigma_3}^2 \mid A$. From here we immediately get that $\overline{\sigma_3}^2 \mid B$, which implies that

$$u = Av_1 + Bv_2 = A'\overline{\sigma_3}^2 v_1 + B'\overline{\sigma_3}^2 v_2$$

for some homogeneous $A'B' \in S(\mathfrak{h}^*)$. As u was arbitrary, it follows that

$$\langle v_1, v_2 \rangle \cap \left\langle \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2} \right\rangle = \left\langle v_1 \overline{\sigma_3}^2, v_2 \overline{\sigma_3}^2 \right\rangle.$$

The submodule $\left\langle v_1 \overline{\sigma_3}^2, v_2 \overline{\sigma_3}^2 \right\rangle$ is isomorphic to $M_{1,c}(\texttt{stand})$ by the same argument that we used for $\langle v_1, v_2 \rangle$. Finally, we get that

$$\begin{aligned}
&\chi_{M_{1,c}(\texttt{stand})/\left\langle v_1, v_2, \overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2}, \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2} \right\rangle} \\
&= \chi_{M_{1,c}(\texttt{stand})} - z^2 \chi_{M_{1,c}(\texttt{stand})} - z^6 \chi_{M_{1,c}(\texttt{stand})} + z^8 \chi_{M_{1,c}(\texttt{stand})} \\
&= \chi_{M_{1,c}(\texttt{stand})} \cdot (1 - z^2)(1 - z^6).
\end{aligned}$$

This proves the claim. □

We have now constructed the module

$$M_{1,c}(\texttt{stand}) / \left\langle v_1, v_2, \overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2}, \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2} \right\rangle$$

and calculated its character. It remains to see that this module is irreducible, which we show by showing it has no singular vectors.

Lemma 8.4.8. *Let $p = 2$, $t = 1$, $c \neq 1$. The module*

$$M_{1,c}(\texttt{stand}) / \left\langle v_1, v_2, \overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2}, \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2} \right\rangle$$

has no singular vectors in odd degrees.

Proof. By Corollary 2.3.5 and Lemma 8.4.1 the action of Ω on $M_{1,c}^k(\texttt{stand})$ is by $c+k$, its action on singular vectors of type `stand` is by c, and its its action on singular vectors of type `triv` is by 0. If k is odd then $c+k \neq c$, and if additionally $c \neq 1$ then $c+k \neq 0$. So, in that case there can not be singular vectors in degree k of $M_{1,c}(\texttt{stand})$. □

It remains to check for singular vectors in even degrees. Let us use Lemma 8.4.1 again. For an even $k \in \mathbb{N}$, the action of Ω on $M_{1,c}^k(\texttt{stand})$ is by the constant $c+k$. The action of Ω on singular vectors of type `triv` is by 0, and whenever $c \neq 0$ we have $c+k \neq 0$, so there can be no singular vectors of type `triv` in even degrees. On the other hand, the action of Ω on singular vectors of type `stand` is by c and for even k we have $c+k=c$, so we need to check even degrees for singular vectors of type `stand`.

Lemma 8.4.9. *Let $p=2$, $t=1$, $c \neq 1$. The module*

$$M_{1,c}(\texttt{stand})/\left\langle v_1, v_2, \overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2}, \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2} \right\rangle$$

has no singular vectors of type `stand` *in degree* 2.

Proof. By Theorem 7.2.9, $M_{1,c}^2(\texttt{stand})$ is a direct sum of two copies of `stand` and an indecomposable extension of two copies of `triv`. The bases for the two copies of `stand` are

$$\{\overline{\sigma_2} \otimes \overline{x_1},\ \overline{\sigma_2} \otimes \overline{x_1}\},$$

$$\{\overline{x_1}^2 \otimes \overline{x_1} + \overline{x_1}^2 \otimes \overline{x_2} + \overline{x_2}^2 \otimes \overline{x_1},\ \overline{x_2}^2 \otimes \overline{x_2} + \overline{x_1}^2 \otimes \overline{x_2} + \overline{x_2}^2 \otimes \overline{x_1}\}.$$

Let us consider just the part of these copies of `stand` corresponding to $\overline{x_1}$ (this is enough as `stand` is irreducible), so the part spanned by

$$\overline{\sigma_2} \otimes \overline{x_1},\ \overline{x_1}^2 \otimes \overline{x_1} + \overline{x_1}^2 \otimes \overline{x_2} + \overline{x_2}^2 \otimes \overline{x_1}.$$

They satisfy

$$D_{y_1-y_3}(\overline{\sigma_2} \otimes \overline{x_1}) = \overline{x_2} \otimes \overline{x_1}$$
$$D_{y_1-y_3}(\overline{x_1}^2 \otimes \overline{x_1} + \overline{x_1}^2 \otimes \overline{x_2} + \overline{x_2}^2 \otimes \overline{x_1}) = c\overline{x_2} \otimes \overline{x_1}.$$

This means that the only singular vector of type `stand` corresponding to $\overline{x_1}$ in $M_{1,c}^2(\texttt{stand})$ is

$$v_1 = c \cdot \overline{\sigma_2} \otimes \overline{x_1} + (\overline{x_1}^2 \otimes \overline{x_1} + \overline{x_1}^2 \otimes \overline{x_2} + \overline{x_2}^2 \otimes \overline{x_1}),$$

and there are no singular vectors in $M_{1,c}^2(\texttt{stand})/\left\langle v_1, v_2, \overline{\sigma_2}^2 \otimes \overline{x_1}, \overline{\sigma_2}^2 \otimes \overline{x_2}, \overline{\sigma_3}^2 \otimes \overline{x_1}, \overline{\sigma_3}^2 \otimes \overline{x_2} \right\rangle$. □

Lemma 8.4.10. *Let $p=2$, $t=1$, $c\neq 1$. The module*

$$M_{1,c}(\texttt{stand})/\left\langle v_1,v_2,\overline{\sigma_2}^2\otimes\overline{x_1},\overline{\sigma_2}^2\otimes\overline{x_2},\overline{\sigma_3}^2\otimes\overline{x_1},\overline{\sigma_3}^2\otimes\overline{x_2}\right\rangle$$

has no singular vectors of type `stand` *in degree* 4.

Proof. By Theorem 7.2.9, $M^4_{1,c}(\texttt{stand})$ is a direct sum of three copies of `stand` and two indecomposable extensions of two copies of `triv`. The bases for the $\overline{x_1}$ part of the copies of `stand` are

$$\overline{\sigma_2}^2\otimes\overline{x_1},\ \overline{\sigma_3}(\overline{x_1}\otimes\overline{x_1}+\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1}),\ \overline{\sigma_2}(\overline{x_1}^2\otimes\overline{x_1}+\overline{x_1}^2\otimes\overline{x_2}+\overline{x_2}^2\otimes\overline{x_1}).$$

Using Theorem 7.2.9 again, we see that the submodule $\langle v_1,v_2\rangle$ which we quotient by contains the following vectors in the `stand` isotypic component of degree 4:

$$\begin{aligned}\sigma_2 v_1 &= c\cdot\overline{\sigma_2}^2\otimes\overline{x_1}+\overline{\sigma_2}(\overline{x_1}^2\otimes\overline{x_1}+\overline{x_1}^2\otimes\overline{x_2}+\overline{x_2}^2\otimes\overline{x_1})\\ \overline{x_1}^2v_1+\overline{x_2}^2v_1+\overline{x_1}^2v_2 &= \overline{\sigma_2}^2\otimes\overline{x_1}+c\cdot\overline{\sigma_2}(\overline{x_1}^2\otimes\overline{x_1}+\overline{x_1}^2\otimes\overline{x_2}+\overline{x_2}^2\otimes\overline{x_1}).\end{aligned}$$

As $c\neq 1$, these two vectors are linearly independent, and so both $\overline{\sigma_2}^2\otimes\overline{x_1}$ and $\overline{\sigma_2}(\overline{x_1}^2\otimes\overline{x_1}+\overline{x_1}^2\otimes\overline{x_2}+\overline{x_2}^2\otimes\overline{x_1})$ are in $\langle v_1,v_2\rangle$, while $\overline{\sigma_3}(\overline{x_1}\otimes\overline{x_1}+\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1})$ is not and spans the $\overline{x_1}$ part of the `stand` isotypic component of $M^4_{1,c}(\texttt{stand})/\left\langle v_1,v_2,\overline{\sigma_j}^2\otimes\overline{x_i}\right\rangle$.

It remains to check if $\overline{\sigma_3}(\overline{x_1}\otimes\overline{x_1}+\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1})$ is singular in $M_{1,c}(\texttt{stand})/\left\langle v_1,v_2,\overline{\sigma_j}^2\otimes\overline{x_i}\right\rangle$. We calculate

$$D_{y_1-y_3}(\overline{\sigma_3}(\overline{x_1}\otimes\overline{x_1}+\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1}))=\overline{x_2}^3\otimes\overline{x_1}+\overline{x_1}^2\overline{x_2}\otimes\overline{x_1}+\overline{x_1}^2\overline{x_2}\otimes\overline{x_2},$$

which can be shown directly or using Lemma 8.4.4 does not lie in the submodule $\langle v_1,v_2\rangle$, so is not zero in the quotient $M_{1,c}(\texttt{stand})/\left\langle v_1,v_2,\overline{\sigma_j}^2\otimes\overline{x_i}\right\rangle$. ∎

Lemma 8.4.11. *Let $p=2$, $t=1$, $c\neq 1$. The module*

$$M_{1,c}(\texttt{stand})/\left\langle v_1,v_2,\overline{\sigma_2}^2\otimes\overline{x_1},\overline{\sigma_2}^2\otimes\overline{x_2},\overline{\sigma_3}^2\otimes\overline{x_1},\overline{\sigma_3}^2\otimes\overline{x_2}\right\rangle$$

has no singular vectors of type `stand` *in degree* 6.

Proof. Using Theorem 7.2.9, the part of the `stand` isotypic component of $M^6_{1,c}(\texttt{stand})$ corresponding to $\overline{x_1}$ has a basis

$$\begin{aligned}A&=\overline{\sigma_2\sigma_3}(\overline{x_1}\otimes\overline{x_1}+\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1})\\ B&=\overline{\sigma_2}^2(\overline{x_1}^2\otimes\overline{x_1}+\overline{x_1}^2\otimes\overline{x_2}+\overline{x_2}^2\otimes\overline{x_1})\\ C&=\overline{\sigma_2}^3\otimes\overline{x_1}\\ D&=\overline{\sigma_3}^2\otimes\overline{x_1}\end{aligned}$$

$$E = \overline{\sigma_3}(\overline{x_1}^3 + \overline{x_1}^2\overline{x_2} + \overline{x_2}^3) \otimes \overline{x_1}.$$

Taking a quotient by $\overline{\sigma_3}^2 \otimes \overline{x_i}$ annihilates

$$\overline{\sigma_3}^2 \otimes \overline{x_1} = D.$$

Taking a quotient by $\langle v_1, v_2 \rangle$ annihilates (again using Theorem 7.2.9)

$$\begin{aligned} \overline{\sigma_2}^2 v_1 &= B + c \cdot C \\ \overline{\sigma_3}(\overline{x_1}v_1 + \overline{x_2}v_1 + \overline{x_1}v_2) &= c \cdot A + E \\ \overline{\sigma_2}(\overline{x_1}^2 v_1 + \overline{x_2}^2 v_1 + \overline{x_1}^2 v_2) &= c \cdot B + C. \end{aligned}$$

So, the resulting basis for the $\overline{x_1}$ part of the copies of $\texttt{stand}$ in $M^6_{1,c}(\texttt{stand})/\left\langle v_1, v_2, \overline{\sigma_j}^2 \otimes \overline{x_i} \right\rangle$ is A. To check that it is not singular in this quotient, we calculate

$$D_{y_1-y_3}(A) = \overline{x_1}^4\overline{x_2} \otimes (\overline{x_1} + \overline{x_2}),$$

which can be shown directly or using Lemma 8.4.4 does not lie in the submodule $\langle v_1, v_2 \rangle$. □

Lemma 8.4.12. *Let $p = 2$, $t = 1$, $c \neq 0, 1$. The quotient of the Verma module $M_{1,c}(\texttt{stand})$ by the submodule generated by v_1, v_2 from Lemma 8.4.2 and $\overline{\sigma_j}^2 \otimes \overline{x_i}$, $j = 2, 3$, $i = 1, 2$ is irreducible, and thus equal to $L_{1,c}(\texttt{stand})$.*

Proof. By Lemma 8.4.8, $M_{1,c}(\texttt{stand})/\left\langle v_1, v_2, \overline{\sigma_j}^2 \otimes \overline{x_i}, j = 2, 3, i = 1, 2 \right\rangle$ has no singular vectors in odd degrees whenever $c \neq 1$. By Lemma 8.4.1, Ω acts on all even degrees of this module by c, on singular vectors of type $\texttt{triv}$ by 0, and on singular vectors of type $\texttt{stand}$ by c. As $c \neq 0$, there are no singular vectors of type $\texttt{triv}$ in even degrees. By Lemmas 8.4.9, 8.4.10 and 8.4.11, there are no singular vectors of type $\texttt{stand}$ in even degrees either. So, $M_{1,c}(\texttt{stand})/\left\langle v_1, v_2, \overline{\sigma_j}^2 \otimes \overline{x_i}, j = 2, 3, i = 1, 2 \right\rangle$ is irreducible.

□

8.5 The irreducible representation $L_{1,c}(\texttt{stand})$ in characteristic 2 for c in $\mathbb{F}_2$

The previous section worked out the characters of $L_{1,c}(\texttt{stand})$ for all $c \neq 0, 1$. For $c = 0$, this character is computed in Lemma 2.6.13. In this section we compute it for $c = 1$.

For the entire section, let $p = 2$, $t = 1$, $c = 1$. The main aim of the section is to prove the following lemma.

Lemma 8.5.1. *Let $p = 2$, $t = 1$, $c = 1$. The irreducible representation $L_{1,1}(\texttt{stand})$ is the quotient of the Verma module $M_{1,1}(\texttt{stand})$ by the submodule generated by the following* *vectors:*

$$\begin{aligned}
v_1 &= \overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1}\\
v_3 &= \overline{\sigma_2}(\overline{x_1}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_2})\\
v_5 &= \overline{\sigma_3}(\overline{x_1}^2\otimes\overline{x_2}+\overline{x_2}^2\otimes\overline{x_1})\\
v_7 &= \overline{\sigma_2\sigma_3}(\overline{x_1}^2\otimes\overline{x_1}+\overline{x_2}^2\otimes\overline{x_1}+\overline{x_2}^2\otimes\overline{x_2}).
\end{aligned}$$

The character of $L_{1,1}(\texttt{stand})$ *is*

$$\chi_{L_{1,1}(\texttt{stand})}(z)=[\texttt{stand}](1+z+z^2+2z^3+z^4+z^5+z^6)+[\texttt{triv}](z+2z^2+2z^4+z^5)$$

and its Hilbert polynomial is

$$Hilb_{L_{1,1}(stand)}(z)=2+3z+4z^2+4z^3+4z^4+3z^5+2z^6=\frac{1-z-z^3-z^5-z^7+2z^8}{(1-z)^2}.$$

By Lemma 8.4.1, the action of Ω on $M_{1,1}^k(\texttt{stand})$ is by $1+k$, the action of Ω on singular vectors of type triv is by 0, and on singular vectors of type stand is by 1. So, we will be looking for singular vectors of type triv in odd degrees and for singular vectors of type stand in even degrees of $M_{1,1}(\texttt{stand})$ and its quotients. We will analyze $M_{1,1}(\texttt{stand})$ and its quotients degree by degree, looking for singular vectors in each degree, taking a quotient by them, and then looking for further singular vectors in this quotient. Note that we need to consider all degrees, as c is not generic there is no guarantee that interesting things happen only in degrees divisible by 2. We will use Theorems 7.2.7 and 7.2.9 significantly and repeatedly to construct bases of the graded pieces of $M_{1,1}(\texttt{stand})$ and its quotients.

Lemma 8.5.2. *Let* $p=2$*. The only singular vector in* $M_{1,1}^1(\texttt{stand})$ *is*

$$v_1=\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1}.$$

Hence, the first terms of the character of $L_{1,1}(\texttt{stand})$ *are*

$$\chi_{L_{1,1}(\texttt{stand})}=[\texttt{stand}](1+z+\ldots)+[\texttt{triv}](z+\ldots).$$

Proof. There are no singular vectors in degree 0 by definition, so $L_{1,1}^0(\texttt{stand})=M_{1,1}^0(\texttt{stand})=\texttt{stand}$. By Theorem 7.2.9 $M_{1,1}^1(\texttt{stand})$ decomposes, as an S_3 representation, into a direct sum of a copy of stand and an indecomposable extension of two copies of triv, with a submodule spanned by $\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1}$ and the quotient spanned by $\overline{x_1}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_2}$. By Lemma 8.4.1 the singular vectors in odd degrees need to span subrepresentations isomorphic to triv. We check:

$$D_{y_1-y_3}(\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1})=1\otimes\overline{x_2}+1\otimes\overline{x_1}+1\otimes\overline{x_2}+1\otimes\overline{x_1}=0$$

$$D_{y_2-y_3}(\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1}) = (12)D_{y_1-y_3}(12)(\overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1}) = 0.$$

So, $v_1 = \overline{x_1}\otimes\overline{x_2}+\overline{x_2}\otimes\overline{x_1}$ is singular. After taking a quotient by the submodule generated by v_1, the vector $\overline{x_1}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_2}$ spans a subrepresentation isomorphic to `triv` in $M^1_{1,1}(\texttt{stand})/\langle v_1\rangle$, so we need to check if it is singular.

$$\begin{aligned} D_{y_1-y_3}(\overline{x_1}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_2}) &= 1\otimes\overline{x_1}+1\otimes\overline{x_2}+1\otimes\overline{x_2}+1\otimes\overline{x_1}+1\otimes\overline{x_1}+1\otimes\overline{x_3}\\ &= 1\otimes\overline{x_2}\neq 0\in M^0_{1,1}(\texttt{stand})/\langle v_1\rangle\,. \end{aligned}$$

This shows v_1 is the only singular vector in degree 1, and proves the lemma. □

We now analyse degree 2 of the quotient $M_{1,1}(\texttt{stand})/\langle v_1\rangle$.

Lemma 8.5.3. *Let $p=2$. There are no singular vectors in $M^2_{1,1}(\texttt{stand})/\langle v_1\rangle$. Hence, the first terms of the character of $L_{1,1}(\texttt{stand})$ are*

$$\chi_{L_{1,1}(\texttt{stand})} = [\texttt{stand}](1+z+z^2\dots)+[\texttt{triv}](z+2z^2\dots).$$

Proof. By Theorem 7.2.9, $M^2_{1,1}(\texttt{stand})$ decomposes as a direct sum of a copy of `stand` and an indecomposable extension of two copies of `triv`. Taking the quotient by v_1, which generates (a priori, a quotient of) $M_{1,1}(\texttt{triv})[-1]$, has the effect in degree 2 of removing one copy of `stand`, leaving $M_{1,1}(\texttt{stand})/\langle v_1\rangle$ with the character stated above.

Using Lemma 8.4.1 we see that we only need to inspect the `stand` isotypic component for singular vectors. The $\overline{x_1}$ components of the two copies of `stand` in $M^2_{1,1}(\texttt{stand})$ are spanned by

$$\overline{\sigma_2}\otimes\overline{x_1},\quad \overline{x_1}^2\otimes\overline{x_1}+\overline{x_1}^2\otimes\overline{x_2}+\overline{x_2}^2\otimes\overline{x_1}.$$

Taking the quotient by v_1 annihilates the vector

$$x_1v_1 = (\overline{\sigma_2}\otimes\overline{x_1})+(\overline{x_1}^2\otimes\overline{x_1}+\overline{x_1}^2\otimes\overline{x_2}+\overline{x_2}^2\otimes\overline{x_1}).$$

So, to see that there are no singular vectors in degree 2, it is enough to check that $\overline{\sigma_2}\otimes\overline{x_1}$ is not singular. We calculate

$$D_{y_1-y_3}(\overline{\sigma_2}\otimes\overline{x_1}) = \overline{x_2}\otimes\overline{x_1}\notin\langle v_1\rangle\,,$$

which proves the lemma. □

Lemma 8.5.4. *Let $p=2$. The only singular vector in $M^3_{1,1}(\texttt{stand})/\langle v_1\rangle$ is*

$$v_3 = \overline{\sigma_2}(\overline{x_1}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_1}+\overline{x_2}\otimes\overline{x_2}).$$

Hence, the first terms of the character of $L_{1,1}(\texttt{stand})$ *are*

$$\chi_{L_{1,1}(\texttt{stand})} = [\texttt{stand}](1 + z + z^2 + 2z^3 + \ldots) + [\texttt{triv}](z + 2z^2 + 0 \cdot z^3 + \ldots).$$

Proof. By Theorem 7.2.9, $M^3_{1,1}(\texttt{stand})$ decomposes as a direct sum of a copy of `stand`, an indecomposable extension of two copies of `stand`, and an indecomposable extension of two copies of `triv`. One can check directly that taking the quotient by $\langle v_1 \rangle$, which is isomorphic to (a priori, a quotient of) $M_{1,1}(\texttt{stand})[-1]$, has the effect of removing one copy of `stand` and one copy of `triv`. Let us do this calculation for `triv`, as `stand` is similar.

By Theorem 7.2.9, the basis for the indecomposable extension of two copies of `triv` is $\overline{\sigma_2}(\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1}) = \sigma_2 v_1$ for the submodule, and $\overline{\sigma_2}(\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2})$ for the quotient. Taking a quotient by $\langle v_1 \rangle$, by Theorem 7.2.7, annihilates $\sigma_2 v_1$ and leaves one copy of `triv` spanned by $\overline{\sigma_2}(\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2})$. After doing a similar computation for `stand`, we conclude that the character of $M^3_{1,1}(\texttt{stand})/\langle v_1 \rangle$ is $[\texttt{stand}]2z^3 + [\texttt{triv}]z^3$.

It remains to check if there are any singular vectors in degree 3. By Lemma 8.4.1 we only need to inspect the `triv` isotypic component, which is 1-dimensional and spanned by $v_3 = \overline{\sigma_2}(\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2})$. We calculate:

$$\begin{aligned}
D_{y_1 - y_3}(v_3) &= \overline{x_2}(\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2}) + \overline{\sigma_2} \otimes \overline{x_1} + \\
&+ \overline{\sigma_2} \otimes \overline{x_1} + \overline{\sigma_2} \otimes \overline{x_1} + \overline{\sigma_2} \otimes \overline{x_1} + \overline{\sigma_2} \otimes \overline{x_2} \\
&= (\overline{x_1 x_2} + \overline{x_2}^2) \otimes \overline{x_1} + (\overline{x_1}^2 + \overline{x_1 x_2}) \otimes \overline{x_2} \\
&= (\overline{x_1} + \overline{x_2}) v_1 \in \langle v_1 \rangle, \\
D_{y_2 - y_3}(v_3) &= s_1 . D_{y_1 - y_3} s_1 . (v_3) \\
&= s_1 . D_{y_1 - y_3}(v_3 + \overline{\sigma_2} v_1) \in \langle v_1 \rangle .
\end{aligned}$$

So, v_3 is singular in $M_{1,1}(\texttt{stand})/\langle v_1 \rangle$ (though not in $M_{1,1}(\texttt{stand})$), and the first terms of the character of $L_{1,1}(\texttt{stand})$ are as stated in the lemma. □

Lemma 8.5.5. *Let* $p = 2$. *There are no singular vectors in* $M^4_{1,1}(\texttt{stand})/\langle v_1, v_3 \rangle$. *Hence, the first terms of the character of* $L_{1,1}(\texttt{stand})$ *are*

$$\chi_{L_{1,1}(\texttt{stand})} = [\texttt{stand}](1 + z + z^2 + 2z^3 + z^4 + \ldots) + [\texttt{triv}](z + 2z^2 + 2z^4 + \ldots).$$

Proof. Using Theorem 7.2.9 again, $M^4_{1,1}(\texttt{stand})$ decomposes as a direct sum of three copies of `stand` and two indecomposable extensions of two copies of `triv`. Using Theorem 7.2.9 one can show directly that taking the quotient by $\langle v_1, v_3 \rangle$ annihilates two copies of `triv` (we skip this computation). Let us do a computation showing what happens to the `stand` isotypic component by taking the quotient, by concentrating on the part of it corresponding to $\overline{x_1}$ under the isomorphism with `stand`.

The basis for the $\overline{x_1}$ part of the `stand` isotypic component of $M^4_{1,1}(\texttt{stand})$ is

$$\begin{aligned}
A &= \overline{\sigma_2}^2 \otimes \overline{x_1} \\
B &= \overline{\sigma_3}(\overline{x_1} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2} + \overline{x_2} \otimes \overline{x_1}) \\
C &= \overline{\sigma_2}(\overline{x_1}^2 \otimes \overline{x_1} + \overline{x_1}^2 \otimes \overline{x_2} + \overline{x_2}^2 \otimes \overline{x_1}).
\end{aligned}$$

Using Theorem 7.2.7 we see that taking the quotient by v_1 annihilates (using Theorem 7.2.7) the following vector in the $\overline{x_1}$ part of the `stand` isotypic component of $M^4_{1,1}(\texttt{stand})$:

$$\overline{\sigma_2 x_1} v_1 = A + C,$$

while taking the quotient by v_3 annihilates

$$\overline{x_1} v_3 = A + \overline{x_2 \sigma_2} v_1 \overset{\mathrm{mod}\ \langle v_1 \rangle}{=} A.$$

So, the $\overline{x_1}$ part of the `stand` isotypic component of $M^4_{1,1}(\texttt{stand})/\langle v_1, v_3 \rangle$ is spanned by B. This shows that the first terms (up to z^4) of the character of $M_{1,1}(\texttt{stand})/\langle v_1, v_3 \rangle$ are as stated in the lemma; it remains only to show that $M^4_{1,1}(\texttt{stand})/\langle v_1, v_3 \rangle$ has no more singular vectors in degree 4.

If it had, they would be of type `stand` by Lemma 8.4.1, so B would be singular. We compute

$$\begin{aligned}
D_{y_1-y_3}(B) &= D_{y_1-y_3}(\overline{\sigma_3}(\overline{x_1} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2} + \overline{x_2} \otimes \overline{x_1})) \\
&\overset{\mathrm{mod}\langle v_1 \rangle}{=} D_{y_1-y_3}(\overline{\sigma_3 x_1} \otimes \overline{x_1}) \\
&= \overline{x_1}^2 \overline{x_2} \otimes \overline{x_1} + \overline{\sigma_3} \otimes \overline{x_2} \notin \langle v_1, v_3 \rangle .
\end{aligned}$$

This proves B is not singular in $M_{1,1}(\texttt{stand})/\langle v_1, v_3 \rangle$. □

Lemma 8.5.6. *Let $p = 2$. The only singular vector in $M^5_{1,1}(\texttt{stand})/\langle v_1, v_3 \rangle$ is*

$$v_5 = \overline{\sigma_3}(\overline{x_1}^2 \otimes \overline{x_2} + \overline{x_2}^2 \otimes \overline{x_1}).$$

Hence, the first terms of the character of $L_{1,1}(\texttt{stand})$ are

$$\chi_{L_{1,1}(\texttt{stand})} = [\texttt{stand}](1 + z + z^2 + 2z^3 + z^4 + z^5 + \ldots) + [\texttt{triv}](z + 2z^2 + 2z^4 + z^5 + \ldots).$$

Proof. We proceed as in the last four lemmas. By Lemma 8.4.1, any singular vectors in degree 5 are of type `triv`, so we only examine this. Using Theorem 7.2.9, a basis for the `triv` component of $M^5_{1,1}(\texttt{stand})$ is

$$A = \overline{\sigma_2}^2(\overline{x_1} \otimes \overline{x_2} + \overline{x_2} \otimes \overline{x_1}) = \overline{\sigma_2}^2 v_1$$

$$B = \overline{\sigma_2}^2(\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2}) = \overline{\sigma_2} v_3$$
$$C = \overline{\sigma_3}(\overline{x_1}^2 \otimes \overline{x_2} + \overline{x_2}^2 \otimes \overline{x_1})$$
$$D = \overline{\sigma_3}(\overline{x_1}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_2}).$$

Here A, C span submodules, and B, D their extensions. Quotienting by $\langle v_1, v_3 \rangle$ annihilates A and B. The only submodule of type `triv` is spanned by C, so we set $v_5 = C$ and check if it is singular:

$$D_{y_1-y_3}(v_5) = \overline{x_2\sigma_2} v_1 \in \langle v_1 \rangle$$
$$D_{y_2-y_3}(v_5) = s_1.\overline{x_2\sigma_2} v_1 = \overline{x_1\sigma_2} v_1 \in \langle v_1 \rangle .$$

This proves the claim that v_5 is singular, and shows that the first terms of the character of $M_{1,1}(\texttt{stand})/\langle v_1, v_3, v_5 \rangle$ are as stated in the lemma. It remains to show that D is not a singular vector, so we calculate

$$D_{y_1-y_3}(D) = \overline{\sigma_2 x_1} \otimes \overline{x_1} + \overline{\sigma_3} v_1 \notin \langle v_1, v_3 \rangle .$$

□

Lemma 8.5.7. *Let $p = 2$. There are no singular vectors in $M^6_{1,1}(\texttt{stand})/\langle v_1, v_3, v_5 \rangle$. Hence, the first terms of the character of $L_{1,1}(\texttt{stand})$ are*

$$\chi_{L_{1,1}(\texttt{stand})} = [\texttt{stand}](1+z+z^2+2z^3+z^4+z^5+z^6+\ldots)+[\texttt{triv}](z+2z^2+2z^4+z^5+0\cdot z^6+\ldots).$$

Proof. We proceed as in the last five lemmas, skipping the `triv` computation and analysing only the $\overline{x_1}$ part of the `stand` component, as by Lemma 8.4.1 any singular vectors in degree 6 are of type `stand`. Using Theorem 7.2.9, a basis for the $\overline{x_1}$ part of the `stand` component component of $M^6_{1,1}(\texttt{stand})$ is

$$A = \overline{\sigma_3}^2 \otimes \overline{x_1}$$
$$B = \overline{\sigma_3}(\overline{x_1}^3 + \overline{x_1}^2\overline{x_2} + \overline{x_2}^3) \otimes \overline{x_1}$$
$$C = \overline{\sigma_2}^3 \otimes \overline{x_1}$$
$$D = \overline{\sigma_2\sigma_3}(\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_1} \otimes \overline{x_2})$$
$$E = \overline{\sigma_2}^2(\overline{x_1}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_1} + \overline{x_1}^2 \otimes \overline{x_2}).$$

Here A, C, D, E span submodules, and B spans an extension of the span of A. By Theorem 7.2.7, taking a quotient by $\langle v_1, v_3 \rangle$ annihilates:

$$\overline{\sigma_2}^2\overline{x_1} v_1 = C + E$$
$$\overline{\sigma_3 x_1}^2 v_1 = B + D + s_1.A$$

$$\begin{aligned} \overline{\sigma_2 x_1} v_3 &= C + s_1.(C+E) \\ \overline{x_1} v_5 &= A + B + D + s_1.A. \end{aligned}$$

Hence, in the quotient we have $A = C = E = B + D = 0$, and we conclude that $L^6_{1,1}(\texttt{stand})$ has at most one copy of $\texttt{stand}$. To see it has exactly one copy, we need to check that B, which spans the $\overline{x_1}$ part of the $\texttt{stand}$ component component of $M^6_{1,1}(\texttt{stand})/\left\langle v_1, v_3, v_5\right\rangle$ is not singular:

$$\begin{aligned} D_{y_1-y_3}(B) &= D_{y_1-y_3}(\overline{\sigma_3}(\overline{x_1}^3 + \overline{x_1}^2\overline{x_2} + \overline{x_2}^3) \otimes \overline{x_1}) \\ &= (\overline{x_1}^4\overline{x_2} + \overline{x_1}^2\overline{x_2}^3 + \overline{x_2}^5) \otimes \overline{x_1} + \overline{\sigma_3 x_1 x_2} \otimes \overline{x_2} \\ &\quad + \overline{\sigma_3}(\overline{x_1}^2 + \overline{x_2}^2 + \overline{x_2 x_3} + \overline{x_3}^2) \otimes \overline{x_1} \\ &= (\overline{x_1}^4\overline{x_2} + \overline{x_1}^2\overline{x_2}^3 + \overline{x_2}^5) \otimes \overline{x_1} + \overline{\sigma_3 x_1 x_2} \otimes \overline{x_2} \\ &\quad + \overline{\sigma_3}(\overline{x_1 x_2} + \overline{x_2}^2) \otimes \overline{x_1} \\ &= (\overline{x_1}^4\overline{x_2} + \overline{x_1}^3\overline{x_2}^2 + \overline{x_1}^2\overline{x_2}^3 + \overline{x_1 x_2}^4 + \overline{x_2}^5) \otimes \overline{x_1} + \overline{\sigma_3 x_1 x_2} \otimes \overline{x_2} \\ &=\notin \left\langle v_1, v_3, v_5\right\rangle. \end{aligned}$$

This proves the claim. □

Lemma 8.5.8. *Let $p = 2$. The only singular vector in $M^7_{1,1}(\texttt{stand})/\left\langle v_1, v_3, v_5\right\rangle$ is*

$$v_7 = \overline{\sigma_2\sigma_3}(\overline{x_1}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_2}).$$

Hence, the first terms of the character of $L_{1,1}(\texttt{stand})$ are

$$\begin{aligned} \chi_{L_{1,1}(\texttt{stand})} &= [\texttt{stand}](1 + z + z^2 + 2z^3 + z^4 + z^5 + z^6 + 0 \cdot z^7 + \ldots) \\ &\quad + [\texttt{triv}](z + 2z^2 + 2z^4 + z^5 + 0 \cdot z^6 + 0 \cdot z^7 + \ldots). \end{aligned}$$

Proof. We proceed as in the last five lemmas. The $\texttt{stand}$ computation, which we skip and which can be done directly using Theorems 7.2.9 and 7.2.7, shows that there are no subrepresentations of type $\texttt{stand}$ in $M^7_{1,1}(\texttt{stand})/\left\langle v_1, v_3, v_5\right\rangle$. To see what the $\texttt{triv}$ component looks like, we use Theorem 7.2.9 to say that the $\texttt{triv}$ component of $M^7_{1,1}(\texttt{stand})$ has a basis:

$$\begin{aligned} A &= \overline{\sigma_2}^3(\overline{x_1} \otimes \overline{x_2} + \overline{x_2} \otimes \overline{x_1}) \\ B &= \overline{\sigma_2}^3(\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2}) \\ C &= \overline{\sigma_3}^2(\overline{x_1} \otimes \overline{x_2} + \overline{x_2} \otimes \overline{x_1}) \\ D &= \overline{\sigma_3}^2(\overline{x_1} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_1} + \overline{x_2} \otimes \overline{x_2}) \\ E &= \overline{\sigma_2\sigma_3}(\overline{x_1}^2 \otimes \overline{x_2} + \overline{x_2}^2 \otimes \overline{x_1}) \\ F &= \overline{\sigma_2\sigma_3}(\overline{x_1}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_2}). \end{aligned}$$

Here A, C, E span submodules and B, D, F their extensions.

By Theorem 7.2.7, taking a quotient by v_1, v_3, v_5 annihilates:

$$\begin{aligned} \overline{\sigma_2}^3 v_1 &= A \\ \overline{\sigma_3}^2 v_1 &= C \\ \overline{\sigma_3}(\overline{x_1}^3 + \overline{x_1}^2\overline{x_2} + \overline{x_2}^2) v_1 &= D + E \\ \overline{\sigma_2}^2 v_3 &= B \\ \overline{\sigma_2} v_5 &= E. \end{aligned}$$

So, $M_{1,1}^7(\texttt{stand})/\langle v_1, v_3, v_5\rangle$ is spanned by

$$v_7 = F = \overline{\sigma_2\sigma_3}(\overline{x_1}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_2}).$$

We calculate

$$\begin{aligned} D_{y_1-y_3}(v_7) &= D_{y_1-y_3}(\overline{\sigma_2\sigma_3}(\overline{x_1}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_1} + \overline{x_2}^2 \otimes \overline{x_2})) \\ &= \dots \\ &= \overline{x_1} v_5 + (\overline{x_1}^2\overline{x_2} + \overline{x_1 x_2}^2 + \overline{x_2}^3) v_3 \in \langle v_1, v_3, v_5\rangle, \end{aligned}$$

and consequently

$$D_{y_2-y_3}(v_7) \in \langle v_1, v_3, v_5\rangle.$$

This proves the lemma. □

Proof of Lemma 8.5.1. In Lemmas 8.5.2, 8.5.3, 8.5.4, 8.5.5, 8.5.6, 8.5.7 and 8.5.8 we looked at degrees $1, 2, \dots, 7$ of the module $M_{1,1}(\texttt{stand})$ and its quotients, looking for singular vectors and taking a quotient by the submodule generated by all the singular vectors found so far. The only singular vectors we found in these successive quotients were v_1, v_3, v_5 and v_7. This proves that there are no more singular vectors in degrees $1, 2, \dots 7$ of the module $M_{1,1}(\texttt{stand})/\langle v_1, v_3, v_5, v_7\rangle$.

By Lemma 8.5.8, $M_{1,1}^k(\texttt{stand})/\langle v_1, v_3, v_5, v_7\rangle = 0$ for $k = 7$ and consequently for all $k \geq 7$. This lets us conclude that $M_{1,1}(\texttt{stand})/\langle v_1, v_3, v_5, v_7\rangle$ has no singular vectors at all (we checked that it has none in degrees up to 7, and it is concentrated in those degrees), so it is equal to the irreducible module $L_{1,1}(\texttt{stand})/\langle v_1, v_3, v_5, v_7\rangle$. Furthermore, it lets us calculate the character of this module by calculating its first seven terms, which we did in Lemma 8.5.8:

$$\chi_{L_{1,1}(\texttt{stand})} = [\texttt{stand}](1 + z + z^2 + 2z^3 + z^4 + z^5 + z^6) + [\texttt{triv}](z + 2z^2 + 2z^4 + z^5).$$

□

Chapter 9

Irreducible Representations of $H_{t,c}(S_3, \mathfrak{h})$ in Characteristic 3

Representation theory of S_3 over an algebraically closed field $\Bbbk$ of characteristic 3 is not semisimple. The irreducible representations are the trivial representation `triv` and the sign representation `sign`; the standard representation `stand` is reducible and indecomposable, with a subrepresentation isomorphic to `triv` and a quotient isomorphic to `sign`. For more detail about these representations, see Section 1.2.

The aim of this chapter is to prove the following theorem.

Theorem 9.0.1. *The characters and Hilbert polynomials of the irreducible representation $L_{t,c}(\tau)$ of the rational Cherednik algebra $H_{t,c}(S_3, \mathfrak{h})$ over an algebraically closed field of characteristic 3, for any t, c and τ, are given by the following tables.*

$p = 3$	$\tau = \texttt{triv}$
$t = 0$, *all c*	$[\texttt{triv}]$
$t = 1$, *all c*	$[\texttt{triv}](1 + z + 2z^2 + z^3 + z^4) + [\texttt{sign}](z + z^2 + z^3)$

$p = 3$	$\tau = \texttt{sign}$
$t = 0$, *all c*	$[\texttt{sign}]$
$t = 1$, *all c*	$[\texttt{sign}](1 + z + 2z^2 + z^3 + z^4) + [\texttt{triv}](z + z^2 + z^3)$

$p = 3$	$\tau = \texttt{triv}$	$\tau = \texttt{sign}$
$t = 0$, *all c*	1	1
$t = 1$, *all c*	$\left(\frac{1 - z^3}{1 - z}\right)^2$ *[DeSu16], Thm 4.1*	$\left(\frac{1 - z^3}{1 - z}\right)^2$

In all cases, we calculate the singular vectors explicitly in the following lemmas.

Proof. The irreducible representation $L_{t,c}(\mathtt{triv})$ is described via its singular vectors, character and Hilbert polynomial in the following Lemmas:

- for $t=0$ and any c, in Lemma 9.1.1;
- for $t=1$ and any c, in Lemma 9.2.1.

The analogous descriptions of the irreducible representation $L_{t,c}(\mathtt{sign})$ can be deduced from the description of $L_{t,-c}(\mathtt{triv})$ and Corollary 2.8.3. Note that the Hilbert polynomials of $L_{1,c}(\mathtt{triv})$ for generic c are also known from Theorem 5.4.1 ([DeSu16], Theorem 4.1). □

9.1 $t = 0$

Lemma 9.1.1. *Let $\Bbbk$ be an algebraically closed field of characteristic 3, and let the values of the parameters be $t=0$ and $c\in\Bbbk$ arbitrary. Vectors $\overline{x_1}$ and $\overline{x_2}$ are singular in the Verma module $M_{0,c}(\mathtt{triv})$. Consequently, the quotient $M_{0,c}(\mathtt{triv})/\langle\overline{x_1},\overline{x_2}\rangle$ is the irreducible representation $L_{0,c}(\mathtt{triv})$ of $H_{0,c}(S_3,\mathfrak{h})$ which is 1-dimensional and concentrated in degree 0, with the character*

$$\chi_{L_{0,c}(\mathtt{triv})}(z)=[\mathtt{triv}]$$

and the Hilbert polynomial

$$Hilb_{L_{0,c}(\mathtt{triv})}(z)=1.$$

Proof. Let us show that $\overline{x_1}$ and $\overline{x_2}$ are singular in $M_{0,c}(\mathtt{triv})$ and the other claims will follow immediately. We compute

$$D_{y_1-y_3}(\overline{x_1})=\left(-c\frac{\mathrm{id}-(12)}{\overline{x_1}-\overline{x_2}}-2c\frac{\mathrm{id}-(13)}{\overline{x_1}-\overline{x_3}}-c\frac{\mathrm{id}-(23)}{\overline{x_2}-\overline{x_3}}\right)(\overline{x_1})=-c-2c-0=0$$
$$D_{y_2-y_3}(\overline{x_1})=\left(c\frac{\mathrm{id}-(12)}{\overline{x_1}-\overline{x_2}}-c\frac{\mathrm{id}-(13)}{\overline{x_1}-\overline{x_3}}-2c\frac{\mathrm{id}-(23)}{\overline{x_2}-\overline{x_3}}\right)(\overline{x_1})=c-c-0=0.$$

By symmetry,

$$D_{y_1-y_3}(\overline{x_2})=(12).D_{y_2-y_3}(\overline{x_1})=0$$
$$D_{y_2-y_3}(\overline{x_2})=(12).D_{y_1-y_3}(\overline{x_1})=0.$$

So, $\overline{x_1}$ and $\overline{x_2}$ are singular. The quotient of $M_{0,c}(\mathtt{triv})\cong S(\mathfrak{h}^*)$ by the submodule generated by these singular vectors is 1-dimensional with the character $\chi_{L_{0,c}(\mathtt{triv})}(z)=[\mathtt{triv}]$ and the Hilbert polynomial $\mathrm{Hilb}_{L_{0,c}(\mathtt{triv})}(z)=1$. □

9.2 $t = 1$

Lemma 9.2.1. *Let $\Bbbk$ be an algebraically closed field of characteristic 3, and let the values of the parameters be $t = 1$ and $c \in \Bbbk$ arbitrary. Vectors $\overline{x_1}^3$ and $\overline{x_2}^3$ are singular in the Verma module $M_{1,c}(\mathtt{triv})$. Consequently, the quotient $M_{1,c}(\mathtt{triv})/\langle \overline{x_1}^3, \overline{x_2}^3 \rangle$ is the irreducible representation $L_{1,c}(\mathtt{triv})$ of $H_{1,c}(S_3, \mathfrak{h})$ which has the character*

$$\chi_{L_{1,c}(\mathtt{triv})}(z) = [\mathtt{triv}](1 + z + 2z^2 + z^3 + z^4) + [\mathtt{sign}](z + z^2 + z^3)$$

and the Hilbert polynomial

$$Hilb_{L_{1,c}(\mathtt{triv})}(z) = 1 + 2z + 3z^2 + 2z^3 + z^4 = \left(\frac{1 - z^3}{1 - z}\right)^2.$$

Proof. Let us first show that $\overline{x_1}^3$ and $\overline{x_2}^3$ are singular in $M_{1,c}(\mathtt{triv})$. We compute

$$\begin{aligned}
D_{y_1-y_3}(\overline{x_1}^3) &= \left(\partial_{y_1-y_3} - c\frac{\mathrm{id} - (12)}{\overline{x_1} - \overline{x_2}} - 2c\frac{\mathrm{id} - (13)}{\overline{x_1} - \overline{x_3}} - c\frac{\mathrm{id} - (23)}{\overline{x_2} - \overline{x_3}}\right)(\overline{x_1}^3) \\
&= -c\left(\overline{x_1}^2 + \overline{x_1 x_2} + \overline{x_2}^2 + 2\overline{x_1}^2 + 2\overline{x_1 x_3} + 2\overline{x_3}^2 + 0\right) \\
&= -c\left(\overline{x_1}^2 + \overline{x_1 x_2} + \overline{x_2}^2 + 2\overline{x_1}^2 - 2\overline{x_1}^2 - 2\overline{x_1 x_2} + 2\overline{x_1}^2 + 4\overline{x_1 x_2} + 2\overline{x_2}^2\right) \\
&= -c\left(3\overline{x_1}^2 + 3\overline{x_1 x_2} + 3\overline{x_2}^2\right) = 0 \\
D_{y_2-y_3}(\overline{x_1}) &= \left(\partial_{y_1-y_3} + c\frac{\mathrm{id} - (12)}{\overline{x_1} - \overline{x_2}} - c\frac{\mathrm{id} - (13)}{\overline{x_1} - \overline{x_3}} - 2c\frac{\mathrm{id} - (23)}{\overline{x_2} - \overline{x_3}}\right)(\overline{x_1}^3) \\
&= c\left(\overline{x_1}^2 + \overline{x_1 x_2} + \overline{x_2}^2 - \overline{x_1}^2 - \overline{x_1 x_3} - \overline{x_3}^2 - 2 \cdot 0\right) \\
&= c\left(\overline{x_1 x_2} + \overline{x_2}^2 + \overline{x_1}^2 + \overline{x_1 x_2} - \overline{x_1}^2 - 2\overline{x_1 x_2} - \overline{x_2}^2\right) = 0.
\end{aligned}$$

By symmetry,

$$\begin{aligned}
D_{y_1-y_3}(\overline{x_2}^3) &= (12).D_{y_2-y_3}(\overline{x_1}^3) = 0 \\
D_{y_2-y_3}(\overline{x_2}^3) &= (12).D_{y_1-y_3}(\overline{x_1}^3) = 0.
\end{aligned}$$

So, $\overline{x_1}^3$ and $\overline{x_2}^3$ are singular.

Consider the quotient $M_{1,c}(\mathtt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3 \right\rangle$ by the submodule generated by singular vectors $\overline{x_1}^3$ and $\overline{x_2}^3$. To calculate its character, note that a basis $a_+ = \overline{x_1} + \overline{x_2}$, $a_- = \overline{x_1} - \overline{x_2}$ of $\mathfrak{h}^*$ respects its decomposition as an indecomposable extension of $\mathtt{sign}$ and $\mathtt{triv}$, with a_- spanning a subrepresentation isomorphic to $\mathtt{sign}$ and a_+ spanning a quotient isomorphic to $\mathtt{triv}$. From here,

$$\left[M^1_{1,c}(\mathtt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3 \right\rangle\right] = [\mathtt{sign}] + [\mathtt{triv}].$$

For degree 2 the basis a_-^2, a_-a_+, a_+^2 tells us that

$$\left[M_{1,c}^2(\texttt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3\right\rangle\right] = [\texttt{sign}] + 2[\texttt{triv}],$$

while for degree 3 the spanning set $a_-^3, a_-^2a_+, a_-a_+^2, a_+^3$ reduces to the basis $a_-^2a_+, a_-a_+^2$ and tells us that

$$\left[M_{1,c}^3(\texttt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3\right\rangle\right] = [\texttt{sign}] + [\texttt{triv}].$$

Finally, in degree 4 the basis $a_-^2a_+^2$ tells us

$$\left[M_{1,c}^4(\texttt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3\right\rangle\right] = [\texttt{triv}].$$

Further degrees are all 0. This shows that the character of $M_{1,c}(\texttt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3\right\rangle$ equals

$$\chi_{M_{1,c}(\texttt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3\right\rangle}(z) = [\texttt{triv}](1+z+2z^2+z^3+z^4) + [\texttt{sign}](z+z^2+z^3)$$

and the Hilbert polynomial is

$$\mathrm{Hilb}_{M_{1,c}(\texttt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3\right\rangle}(z) = 1+2z+3z^2+2z^3+z^4 = \left(\frac{1-z^3}{1-z}\right)^2.$$

It remains to show that $M_{1,c}(\texttt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3\right\rangle$ is irreducible. We do so using a method similar to the proof of Lemma 8.2.2, using the fact that the module $M_{1,c}(\texttt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3\right\rangle \cong S(\mathfrak{h}^*)/\left\langle \overline{x_1}^3, \overline{x_2}^3\right\rangle$ is also a Frobenius algebra. We work with the basis $\{\overline{x_1}^i\overline{x_2}^j \mid 0 \le i,j < 3\}$.

Assume that U is a nonzero graded submodule of $M_{1,c}(\texttt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3\right\rangle$, and $u \in U$ some nonzero homogeneous vector. Multiplying by a nonzero constant if needed, u can be written as

$$u = \overline{x_1}^a\overline{x_2}^b + \sum_{a'>a} \alpha_{a'}\overline{x_1}^{a'}\overline{x_2}^{a+b-a'}$$

for some a, b with $0 \le a, b < 3$ and $\alpha_{a'} \in \Bbbk$. As U is a subrepresentation, the vector

$$\overline{x_1}^{2-a}\overline{x_2}^{2-b}u = \overline{x_1}^2\overline{x_2}^2 + \sum_{a'>a} \alpha_{a'}\overline{x_1}^{2+a'-a}\overline{x_2}^{2+a-a'} = \overline{x_1}^2\overline{x_2}^2$$

is also in U (this is using $\overline{x_1}^3 = 0$).

So, any nonzero graded submodule U of $M_{1,c}(\texttt{triv})/\left\langle \overline{x_1}^3, \overline{x_2}^3\right\rangle$ contains its top degree which is spanned by $\overline{x_1}^2\overline{x_2}^2$. As it is a submodule, it also contains

$$(y_1-y_2)^2(y_1+y_2+y_3)^2(\overline{x_1}^2\overline{x_2}^2) \in U.$$

We now calculate this vector.

First, note that it makes sense, because in characteristic 3 we have

$$\begin{aligned}(y_1-y_3)+(y_2-y_3) &= y_1+y_2-2y_3\\ &= y_1+y_2+y_3=Y\end{aligned}$$

hence $Y\in\mathfrak{h}$. It satisfies $\langle Y,\alpha_s\rangle=0$ for all α_s, so $D_Y=\partial_Y$. So, it is easy to calculate

$$\begin{aligned}D^2_{y_1-y_2}D^2_{y_1+y_2+y_3}(\overline{x_1}^2\overline{x_2}^2) &= 2D^2_{y_1-y_2}D_{y_1+y_2+y_3}(\overline{x_1}^2\overline{x_2}+\overline{x_1}\overline{x_2}^2)\\ &= 2D^2_{y_1-y_2}\left(2\overline{x_1}\overline{x_2}+\overline{x_1}^2+\overline{x_2}^2+2\overline{x_1}\overline{x_2}\right)\\ &= 2D^2_{y_1-y_2}\left(\overline{x_1}^2+\overline{x_1}\overline{x_2}+\overline{x_2}^2\right).\end{aligned}$$

We now use the fact that in characteristic 3,

$$\overline{\sigma_2}=2(\overline{x_1}^2+\overline{x_1}\overline{x_2}+\overline{x_2}^2),$$

so on this vector $D_{y_1-y_2}=\partial_{y_1-y_2}$, to get

$$\begin{aligned}D^2_{y_1-y_2}D^2_{y_1+y_2+y_3}(\overline{x_1}^2\overline{x_2}^2) &= 2D_{y_1-y_2}\partial_{y_1-y_2}\left(\overline{x_1}^2+\overline{x_1}\overline{x_2}+\overline{x_2}^2\right)\\ &= 2D_{y_1-y_2}\left(2\overline{x_1}+\overline{x_2}-\overline{x_1}-2\overline{x_2}\right)\\ &= 2D_{y_1-y_2}\left(\overline{x_1}-\overline{x_2}\right)\\ &= 2\left(\partial_{y_1-y_2}-2c\frac{\mathrm{id}-(12)}{\overline{x_1}-\overline{x_2}}-c\frac{\mathrm{id}-(13)}{\overline{x_1}-\overline{x_3}}+c\frac{\mathrm{id}-(23)}{\overline{x_2}-\overline{x_3}}\right)(\overline{x_1}-\overline{x_2})\\ &= 2\left(1+1-2c\cdot 2-c\cdot 1+c\cdot(-1)\right)\\ &= 4-12c=1.\end{aligned}$$

This calculation lets us conclude that $1\in U$. However, 1 generates the entire module $M_{1,c}(\mathtt{triv})/\left\langle\overline{x_1}^3,\overline{x_2}^3\right\rangle$, so we conclude $U=M_{1,c}(\mathtt{triv})/\left\langle\overline{x_1}^3,\overline{x_2}^3\right\rangle$. This means we just showed that every nonzero graded submodule of $M_{1,c}(\mathtt{triv})/\left\langle\overline{x_1}^3,\overline{x_2}^3\right\rangle$ is equal to the whole module, so $M_{1,c}(\mathtt{triv})/\left\langle\overline{x_1}^3,\overline{x_2}^3\right\rangle$ is irreducible and equal to $L_{1,c}(\mathtt{triv})$. □

Chapter 10

Auxiliary Computations in the Rescaled Young Basis

For the rest of the thesis, the characteristic of the field $\Bbbk$ is $p > 3$. Therefore the category of representations of S_3 is semisimple and, as explained in Section 3.2, we can realise $\mathfrak{h}^*$ as a subrepresentation of V^*, with the rescaled Young basis

$$b_+ = x_1 + x_2 - 2x_3, \quad b_- = x_1 - x_2$$

from Section 7.1. We will be using the bases of $M_{t,c}(\tau)$ from Theorems 7.2.2 and 7.2.4, but to manipulate those bases we will first need to know the action of all reflections on b_+, b_-, which can be easily calculated to be:

$$\begin{aligned} (12).b_+ &= b_+ & (12).b_- &= -b_- \\ (13).b_+ &= \frac{-b_+ - 3b_-}{2} & (13).b_- &= \frac{-b_+ + b_-}{2} \\ (23).b_+ &= \frac{-b_+ + 3b_-}{2} & (23).b_- &= \frac{b_+ + b_-}{2}. \end{aligned} \tag{10.0.1}$$

Lemma 10.0.2. *If f is a singular vector in a representation of $H_{1,c}(S_3, \mathfrak{h})$ over a field of characteristic $p > 3$, and an element of an irreducible S_3 subrepresentation τ, then*

$$\Omega.f = \begin{cases} 0 & \tau = \mathtt{triv}, \\ 6cf & \tau = \mathtt{sign} \\ 3cf & \tau = \mathtt{stand}. \end{cases}$$

Proof. As $y.f = 0$ for all $y \in \mathfrak{h}$, $\Omega.f = \sum_{s \in S} c(1 - s).f$ and a simple calculation gives the stated scalars. □

The main advantages of the basis b_+, b_- for $\mathfrak{h}^*$ are two-fold. Firstly, the rescaled Young basis allows us to easily compute bases for $S(\mathfrak{h}^*)$ and for $M_{t,c}(\tau)$ which are compatible with their decomposition as S_3 representations, thus reducing the space in which we need to look for singular vectors using Lemma 10.0.2. Secondly, calculating Dunkl operators in the rescaled Young basis is more manageable than in the monomial basis. To elaborate the second point, basis vectors from Theorems 7.2.2 and 7.2.4 are all a product of a large S_3 invariant factor which behaves very well under Dunkl operators and a small number of low degree expressions in b_+, b_-. Applying the Dunkl operators to such a basis vector v will result in a sum of finitely many terms from this basis, and the number of terms will not depend on the graded piece v is in, see for example Lemma 11.2.1. The main disadvantage of the basis b_+, b_- and resulting bases for $S(\mathfrak{h}^*)$ and for $M_{t,c}(\tau)$ is that multiplication (except by a symmetric polynomial) in these bases is more involved than in the monomial basis.

In Proposition 3.2.2, we defined a map $\pi : \mathfrak{h}^* \to V^*$ by $\pi(\overline{x_i}) = x_i - \frac{X}{n}$ for all $i \in \{1, \ldots, n\}$ and this map extends naturally to a map $S(\mathfrak{h}^*) \to S(V^*)$. In Proposition 3.3.12 we define $\sigma_i = \pi(\overline{\sigma_i})$ for all $i \in \{2, \ldots, n\}$, where $\overline{\sigma_i}$ is the elementary symmetric polynomial $\widetilde{\sigma}_i$ of degree i, under the induced quotient map $S(V^*) \to S(\mathfrak{h}^*)$. Therefore

$$\begin{aligned}\sigma_2 &= \pi(-\overline{x_1}^2 - \overline{x_1 x_2} - \overline{x_2}^2) \\ &= \frac{1}{3}\big(x_1x_2 + x_1x_3 + x_2x_3\big) - \frac{1}{3}\big(x_1^2 + x_2^2 + x_3^2\big),\end{aligned}$$

$$\begin{aligned}\sigma_3 &= \pi(-\overline{x_1}^2\overline{x_2} - \overline{x_1 x_2}^2) \\ &= \frac{-1}{3^2}\big(x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2\big) \\ &\quad + \frac{2}{3^3}\big(x_1^3 + x_2^3 + x_3^3\big) + \frac{2^2}{3^2}\big(x_1x_2x_3\big).\end{aligned}$$

One can verify that in the bases $\{b_+^2, b_+b_-, b_-^2\}$ for $S^2(\mathfrak{h}^*)$ and $\{b_+^3, b_+^2b_-, b_+b_-^2, b_-^3\}$ for $S^3(\mathfrak{h})$, these expressions become

$$\sigma_2 = \frac{-1}{2^2 \cdot 3}b_+^2 + \frac{-1}{2^2}b_-^2,$$

$$\sigma_3 = \frac{-1}{2^2 \cdot 3^3}b_+^3 + \frac{1}{2^2 \cdot 3}b_+b_-^2.$$

Additionally, in Theorem 7.2.2 we defined

$$q = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = x_1^2x_2 + x_1x_3^3 + x_2^2x_3 - x_1^2x_3 - x_1x_2^2 - x_2x_3^2$$

which in the basis $\{b_+^3, b_+^2b_-, b_+b_-^2, b_-^3\}$ for $S^3(\mathfrak{h})$ becomes

$$q = \frac{1}{2^2}b_+^2b_- + \frac{1}{2^2}b_-^3.$$

We gather all tedious multiplication formulas among b_+, b_-, $(-b_+^2 + 3b_-^2)$, $2b_+b_-$ and q

we will need for our computations of singular vectors later into the following lemma.

Lemma 10.0.3. *The following identities hold in* $S(\mathfrak{h}^*)$*:*

$$\begin{aligned}
b_+^2 &= -6\sigma_2 - \frac{1}{2}(-b_+^2 + 3b_-^2)\\
b_-^2 &= -2\sigma_2 + \frac{1}{6}(-b_+^2 + 3b_-^2)\\
b_+ \cdot 2b_+b_- &= 6 \cdot q - 6 \cdot \sigma_2 b_-\\
b_+ \cdot (-b_+^2 + 3b_-^2) &= 54 \cdot \sigma_3 + 6 \cdot \sigma_2 b_+\\
b_- \cdot 2b_+b_- &= 18 \cdot \sigma_3 - 2 \cdot \sigma_2 b_+\\
b_- \cdot (-b_+^2 + 3b_-^2) &= -6 \cdot q - 6 \cdot \sigma_2 b_-\\
(-b_+^2 + 3b_-^2)^2 &= 72 \cdot \sigma_2^2 - 108 \cdot \sigma_3 b_+ - 6 \cdot \sigma_2(-b_+^2 + 3b_-^2)\\
(-b_+^2 + 3b_-^2) \cdot 2b_+b_- &= 108 \cdot \sigma_3 b_- + 6 \cdot \sigma_2 \cdot 2b_+b_-\\
(2b_+b_-)^2 &= 24 \cdot \sigma_2^2 + 36 \cdot \sigma_3 b_+ + 2 \cdot \sigma_2(-b_+^2 + 3b_-^2)\\
q \cdot b_+ &= -9 \cdot \sigma_3 b_- - 1 \cdot \sigma_2 \cdot 2b_+b_-\\
q \cdot b_- &= 3 \cdot \sigma_3 b_+ + \frac{1}{3} \cdot \sigma_2(-b_+^2 + 3b_-^2)\\
(-b_+^2 + 3b_-^2) \cdot q &= -12 \cdot \sigma_2^2 b_- + 9 \cdot \sigma_3 \cdot 2b_+b_-\\
2b_+b_- \cdot q &= 4 \cdot \sigma_2^2 b_+ - 3 \cdot \sigma_3(-b_+^2 + 3b_-^2)\\
q^2 &= -27\sigma_3^2 - 4\sigma_2^3.
\end{aligned}$$

Proof. We shall prove the first of these identities, and the rest are proved similarly by direct computation. We have

$$\begin{aligned}
6\sigma_2 &= \frac{-2 \cdot 3}{2^2 \cdot 3} b_+^2 + \frac{-2 \cdot 3}{2^2} b_-^2\\
&= \frac{-1}{2} b_+^2 - \frac{3}{2} b_-^2\\
&= \frac{-1}{2}\big(-b_+^2 + 3b_-^2\big) - b_+^2
\end{aligned}$$

hence $b_+^2 = -6\sigma_2 + \frac{-1}{2}\big(-b_+^2 + 3b_-^2\big)$. □

In characteristic $p \nmid 3$, to determine whether a homogenous vector is singular in $M_{t,c}(S_3, \mathfrak{h}, \tau)$ it is sufficient to check that it spans a representation of S_2 and is in the kernel of the Dunkl operator D_{y_1} as shown in the following lemma. Since y_1 is not an element of $\mathfrak{h}$ it may seem improper to consider the Dunkl operator D_{y_1} acting on $M_{t,c}(S_3, \mathfrak{h}, \tau)$. However, $M_{t,c}(S_3, V, \tau) \cong S(V^*) \otimes \tau$ is a Verma module for the rational Cherednik algebra $H_{t,c}(S_n, V)$ and $S(\mathfrak{h}^*) \otimes \tau$ is a subspace of $M_{t,c}(S_3, V, \tau)$. We shall see that this subspace is preserved by all Dunkl operators, thus $S(\mathfrak{h}^*) \otimes \tau \cong M_{t,c}(S_3, \mathfrak{h}, \tau)$ is a module for $H_{t,c}(S_3, V, \tau)$ and we can consider the Dunkl operators D_{y_1} as acting on elements of $M_{t,c}(S_3, \mathfrak{h}, \tau)$.

Lemma 10.0.4. *Suppose that* $p \nmid 3$. *Let* $f \in M_{t,c}(S_3, \mathfrak{h}, \tau)$*, or any of its quotients, be homogeneous and suppose* $(12).f = \pm f$. *For all* i, j*, the vector* f *satisfies* $D_{y_i - y_j}(f) = 0$ *if and only if* $D_{y_1}(f) = 0$.

Proof. As shown in Lemma 4.2.3, $Y = y_1 + y_2 + y_3$ commutes with all $x \in \mathfrak{h}^*$ and therefore commutes with all $f_i \in S(\mathfrak{h}^*)$. Therefore for any tensor $\sum f_i \otimes v_i \in S(\mathfrak{h}^*) \otimes \tau \cong M_{t,c}(S_3, \mathfrak{h}, \tau)$ we have

$$D_Y\left(\sum f_i \otimes v_i\right) = Y\left(\sum f_i \otimes v_i\right) = \sum Y f_i \otimes v_i = \sum f_i Y \otimes v_i = \sum f_i \otimes Y v_i = 0,$$

hence $D_Y = 0$ as an operator on $M_{t,c}(S_3, \mathfrak{h}, \tau)$.

($\Longrightarrow$) Suppose $f \in M_{t,c}(\tau, S_3, \mathfrak{h})$ or one of its quotients is singular for $H_{t,c}(S_3, \mathfrak{h})$, so $D_{y_i - y_j}(f) = 0$ for all i, j. We now have

$$D_{y_1}(f) = \frac{1}{3}\big(D_{y_1 - y_2}(f) + D_{y_1 - y_3}(f) + D_{y_1 + y_2 + y_3}(f)\big) = 0$$

as claimed. Notice that this argument fails precisely in characteristic $p \mid 3$.

($\Longleftarrow$) Suppose $f \in M_{t,c}(S_3, \mathfrak{h}, \tau)$ or one of its quotients satisfies $D_{y_1}(f) = 0$ and $(12).f = \pm f$. Now

$$D_{y_2}(f) = (12) D_{y_1} (12)(f) = \pm (12) D_{y_1}(f) = 0.$$

Furthermore,

$$D_{y_3}(f) = D_{y_1 + y_2 + y_3}(f) - D_{y_1}(f) - D_{y_2}(f) = 0,$$

so $D_y(f) = 0$ for all $y \in V$ and in particular all $y \in \mathfrak{h}$. □

When looking for singular vectors, we will make liberal use of Lemma 10.0.2 to reduce the space where we are looking for singular vectors, and Lemma 10.0.4 to reduce the task of finding $\cap_{i,j} \ker D_{y_i - y_j}$ to the task of finding $\ker D_{y_1}$. For that purpose, let us make D_{y_1} explicit as

$$D_{y_1} = t\partial_{y_1} \otimes \mathrm{id} - c\frac{\mathrm{id} - (12)}{x_1 - x_2} \otimes (12) - c\frac{\mathrm{id} - (13)}{x_1 - x_3} \otimes (13).$$

Let us note that for any reflection $s \in S$, any vector $v \in M_{t,c}(\tau)$ or its quotient, and any $f \in S(\mathfrak{h}^*)^{S_3}$ we have

$$\left(\frac{\mathrm{id} - s}{\alpha_s} \otimes s\right)(fv) = f \cdot \left(\frac{\mathrm{id} - s}{\alpha_s} \otimes s\right)(v). \tag{10.0.5}$$

This makes the calculation of Dunkl operators on our chosen basis of $M_{t,c}(\tau)$ manageable. Let us now calculate the action of the divided difference operators and the partial derivatives on all factors which appear in the bases of $S(\mathfrak{h}^*)$ and for $M_{t,c}(\tau)$ from Theorems 7.2.2 and 7.2.4.

Lemma 10.0.6. *The following identities hold in $S(\mathfrak{h}^*)$:*

$$\begin{aligned}
\frac{\mathrm{id}-(12)}{x_1-x_2}(b_+) &= 0, & \frac{\mathrm{id}-(13)}{x_1-x_3}(b_+) &= 3,\\
\frac{\mathrm{id}-(12)}{x_1-x_2}(b_-) &= 2, & \frac{\mathrm{id}-(13)}{x_1-x_3}(b_-) &= 1,\\
\frac{\mathrm{id}-(12)}{x_1-x_2}(-b_+^2+3b_-^2) &= 0, & \frac{\mathrm{id}-(13)}{x_1-x_3}(-b_+^2+3b_-^2) &= 3(-b_++3b_-),\\
\frac{\mathrm{id}-(12)}{x_1-x_2}(2b_+b_-) &= 4b_+, & \frac{\mathrm{id}-(13)}{x_1-x_3}(2b_+b_-) &= -b_++3b_-,\\
\frac{\mathrm{id}-(12)}{x_1-x_2}(q) &= -2\sigma_2-\frac{1}{3}(-b_+^2+3b_-^2), & \frac{\mathrm{id}-(13)}{x_1-x_3}(q) &= 2\sigma_2-\frac{1}{6}(-b_+^2+3b_-^2)+\frac{1}{2}\cdot 2b_+b_-.
\end{aligned}$$

Proof. Direct computation. □

Lemma 10.0.7. *The following identities hold in $S(\mathfrak{h}^*)$:*

$$\begin{aligned}
\partial_{y_1}(b_+) &= 1, & \partial_{y_1}(b_-) &= 1,\\
\partial_{y_1}(-b_+^2+3b_-^2) &= -2b_++6b_-, & \partial_{y_1}(2b_+b_-) &= 2b_++2b_-\\
\partial_{y_1}(\sigma_2) &= \frac{-1}{6}b_++\frac{-1}{2}b_-, & \partial_{y_1}(\sigma_3) &= \frac{1}{36}(-b_+^2+3b_-^2)+\frac{1}{12}(2b_+b_-),\\
\partial_{y_1}(q) &= \frac{-1}{4}(-b_+^2+3b_-^2)+\frac{1}{4}(2b_+b_-).
\end{aligned}$$

Proof. Direct computation. □

Finally, in the attempt to express the value of the Dunkl operator D_{y_1} on a vector in our basis as a linear combination of basis vectors, we will often need the following computations.

Lemma 10.0.8. *The following identities hold in $S(\mathfrak{h}^*)$:*

$$\begin{aligned}
\partial_{y_1}(\sigma_2)\cdot b_+ &= \sigma_2+\frac{1}{12}(-b_+^2+3b_-^2)+\frac{-1}{4}(2b_+b_-)\\
\partial_{y_1}(\sigma_3)\cdot b_+ &= \frac{3}{2}\sigma_3+\frac{1}{6}\sigma_2b_+-\frac{1}{2}\sigma_2b_-+\frac{1}{2}q\\
\partial_{y_1}(\sigma_2)\cdot b_- &= \sigma_2+\frac{-1}{12}(-b_+^2+3b_-^2)+\frac{-1}{12}(2b_+b_-)\\
\partial_{y_1}(\sigma_3)\cdot b_- &= \frac{3}{2}\sigma_3+\frac{-1}{6}\sigma_2b_++\frac{-1}{6}\sigma_2b_-+\frac{-1}{6}q\\
\partial_{y_1}(\sigma_2)\cdot(-b_+^2+3b_-^2) &= -9\sigma_3-\sigma_2b_++3\sigma_2b_-+3q\\
\partial_{y_1}(\sigma_3)\cdot(-b_+^2+3b_-^2) &= 2\sigma_2^2-3\sigma_3b_++9\sigma_3b_--\frac{1}{6}\sigma_2(-b_+^2+3b_-^2)+\frac{1}{2}\sigma_2(2b_+b_-)\\
\partial_{y_1}(\sigma_2)\cdot 2b_+b_- &= -9\sigma_3+\sigma_2b_++\sigma_2b_--q\\
\partial_{y_1}(\sigma_3)\cdot 2b_+b_- &= 2\sigma_2^2+3\sigma_3b_++3\sigma_3b_-+\frac{1}{6}\sigma_2(-b_+^2+3b_-^2)+\frac{1}{6}\sigma_2(2b_+b_-)\\
\partial_{y_1}(\sigma_2)\cdot q &= \frac{-3}{2}\sigma_3b_++\frac{3}{2}\sigma_3b_-+\frac{-1}{6}\sigma_2(-b_+^2+3b_-^2)+\frac{1}{6}\sigma_2(2b_+b_-)
\end{aligned}$$

$$\partial_{y_1}(\sigma_3) \cdot q = \frac{1}{3}\sigma_2^2 b_+ \frac{-1}{3}\sigma_2^2 b_- + \frac{-1}{4}\sigma_3(-b_+^2 + 3b_-^2) + \frac{1}{4}\sigma_3(2b_+ b_-).$$

Proof. Direct computation. □

Chapter 11

Irreducible Representations of $H_{t,c}(S_3, \mathfrak{h})$ in Characteristic $p > 3$ for Generic c

The aim of this chapter is to prove the following theorem.

Theorem 11.0.1. *The characters and the Hilbert polynomials of the irreducible representation $L_{t,c}(\tau)$ of the rational Cherednik algebra $H_{t,c}(S_3, \mathfrak{h})$ over an algebraically closed field of characteristic $p > 3$, for generic c, $t = 0, 1$, and any τ, are given by the following tables.*

Characters:

$p > 3$	$t = 0$, $c \neq 0$ *generic*	$t = 1$, $c \notin \mathbb{F}_p$
$\tau = \texttt{triv}$	$[\texttt{triv}] + [\texttt{stand}](z + z^2) + [\texttt{sign}]z^3$ *[DeSa14], Prop 4.1*	$\chi_{S(\mathfrak{h}^*)}(z) \cdot (1 - z^{2p})(1 - z^{3p})$ *[DeSa14], Prop 4.2*
$\tau = \texttt{sign}$	$[\texttt{sign}] + [\texttt{stand}](z + z^2) + [\texttt{triv}]z^3$ *[DeSa14], Prop 4.1*	$\chi_{S(\mathfrak{h}^*)}(z) \cdot [\texttt{sign}](1 - z^{2p})(1 - z^{3p})$ *[DeSa14], Prop 4.2*
$\tau = \texttt{stand}$	$[\texttt{stand}] + ([\texttt{triv}] + [\texttt{sign}])z + [\texttt{stand}]z^3$ *[DeSa14], Prop 4.1*	$\chi_{S(\mathfrak{h}^*)}(z) \cdot [\texttt{stand}](1 - z^{p})(1 - z^{3p})$ *[DeSa14], Prop 4.2*

where

$$\chi_{S(\mathfrak{h}^*)}(z) = \frac{1}{(1 - z^2)(1 - z^3)} \left([\texttt{triv}] + [\texttt{stand}](z + z^2) + [\texttt{sign}]z^3\right).$$

Hilbert polynomials:

$p > 3$	$t = 0$, $c \neq 0$ *generic*	$t = 1$, $c \notin \mathbb{F}_p$
$\tau = \texttt{triv}$	$1 + 2(z + z^2) + z^3$ *[DeSa14], Prop 4.1*	$\dfrac{(1 - z^{2p})(1 - z^{3p})}{(1 - z)^2}$ *[DeSa14], Prop 4.2*
$\tau = \texttt{sign}$	$1 + 2(z + z^2) + z^3$ *[DeSa14], Prop 4.1*	$\dfrac{(1 - z^{2p})(1 - z^{3p})}{(1 - z)^2}$ *[DeSa14], Prop 4.2*
$\tau = \texttt{stand}$	$2 + 2z + 2z^2$ *[DeSa14], Prop 4.1*	$\dfrac{2(1 - z^{p})(1 - z^{3p})}{(1 - z)^2}$ *[DeSa14], Prop 4.2*

In all cases, the singular vectors are known and described in the proof.

We note that [DeSa14] already provides all of these characters and Hilbert polynomials, not just in the case of S_3 but in the much greater generality of S_n. We use their results to conclude that the modules we construct are indeed irreducible. Our main contribution is to give, in the case of S_3, explicit singular vectors. Other than for its general interest, our methods are used again in the next chapter to describe $L_{t,c}(\tau)$ for special c, which is a case [DeSa14] does not discuss, as their methods are geometric and use the Calogero-Moser space which is smooth precisely for generic c.

Proof of Theorem 11.0.1. The generic values of c ($c \neq 0$ for $t = 0$ and $c \notin \mathbb{F}_p$ for $t = 1$) are given in Proposition 4.1.3. For those values, the characters and Hilbert polynomials are given by [DeSa14] in Propositions 4.1 and 4.2 of their paper [DeSa14], as discussed in Proposition 5.3.1 and Proposition 5.3.2 of Section 5.3.

For $t = 0, 1$ and $\tau = \texttt{triv}, \texttt{sign}$, the Hilbert polynomial of $L_{t,c}(\tau)$ for generic c from Proposition 5.3.1 and Proposition 5.3.2 coincides with the Hilbert polynomials of $N_{t,c}(\tau)$ from Examples 4.1.1 and 4.1.2, so we conclude that $L_{t,c}(\tau) = N_{t,c}(\tau)$. In this case all the singular vectors are known; for $t = 0$ they are $\sigma_i \otimes v$ for $i = 2, 3$ and $v \in \tau$, and for $t = 1$ they are $\sigma_i^p \otimes v$ for $i = 2, 3$ and $v \in \tau$.

For $t = 0, 1$ and $\tau = \texttt{stand}$, comparing the Hilbert polynomials of $L_{t,c}(\texttt{stand})$ for generic c from Proposition 5.3.1 and Proposition 5.3.2 with the Hilbert polynomials of of $N_{t,c}(\texttt{stand})$ from Examples 4.1.1 and 4.1.2 shows that $L_{t,c}(\tau)$ is a proper quotient of $N_{t,c}(\tau)$. For $t = 0$ the singular vectors are computed in Lemma 11.1.1, and alternatively are available in a different basis in Corollary 8.3.5. For $t = 1$ the singular vectors are computed in Lemma 11.2.8, and the Hilbert polynomial of the quotient of $N_{1,c}(\texttt{stand})$ by these singular vectors is computed in Lemma 11.2.19. Observing this polynomial is equal to the Hilbert polynomial of $L_{1,c}(\texttt{stand})$, we conclude this quotient is irreducible. Its character is then straightforward to compute. □

11.1 The irreducible representation $L_{0,c}(\texttt{stand})$ characteristic $p > 3$ for generic c

The only remaining task for $t = 0$ is to describe explicitly the singular vectors in $M_{0,c}(\texttt{stand})$ which is done in the following lemma. However, the result already appears in Section 8.3 but the approach is slightly different because in characteristic 2 the results of [DeSa14] do not apply, and irreducibility must be shown directly. Otherwise the approaches are similar by considering each degree in turn, although different bases are used.

Lemma 11.1.1. *Let $\Bbbk$ be an algebraically closed field of characteristic $p > 3$, and let the values of the parameters be $t = 0$ and $c \neq 0$. The irreducible representation $L_{0,c}(\texttt{stand})$ of $H_{0,c}(S_3, \mathfrak{h})$ is equal to the quotient of the Verma module $M_{0,c}(\texttt{stand})$ by the singular vectors*

$$
\begin{aligned}
v_+ &= -b_+ \otimes b_+ + 3b_- \otimes b_- \\
v_- &= b_+ \otimes b_- + b_- \otimes b_+ \\
&\sigma_3 \otimes b_+ \\
&\sigma_3 \otimes b_-.
\end{aligned}
$$

It has the character

$$\chi_{L_{0,c}(\texttt{stand})}(z) = [\texttt{stand}] + ([\texttt{triv}] + [\texttt{sign}])z + [\texttt{stand}]z^2$$

and the Hilbert polynomial

$$Hilb_{L_{0,c}(\texttt{stand})}(z) = 2 + 2z + 2z^2.$$

Proof. Let us check that the stated vectors are indeed singular. The vectors $\sigma_3 \otimes b_+$ and $\sigma_3 \otimes b_-$ are always singular, and span an S_3 subrepresentation isomorphic to stand in degree 3. By Theorem 7.2.4, v_+ and v_- span an S_3 subrepresentation isomorphic to stand in degree 1. Using the computations in Chapter 10 we calculate

$$
\begin{aligned}
D_{y_1}(v_+) &= -c\left(\frac{\mathrm{id} - (12)}{x_1 - x_2} \otimes (12) + \frac{\mathrm{id} - (13)}{x_1 - x_3} \otimes (13)\right)(-b_+ \otimes b_+ + 3b_- \otimes b_-) \\
&= -c\left(0 + 6 \otimes b_- + 3 \otimes \frac{-b_+ - 3b_-}{2} - 3 \otimes \frac{-b_+ + b_-}{2}\right) = 0.
\end{aligned}
$$

By Lemma 10.0.4, since $D_{y_1}(v_+) = 0$ and $(12).v_+ = v_+$ we can conclude that v_+ is singular. Now

$$v_- = \frac{2}{3}\left((23) + \frac{1}{2}\right)v_+$$

so it follows that v_- is also singular.

We now will calculate the character of $M_{0,c}(\texttt{stand})/\langle v_\pm, \sigma_3 \otimes b_\pm \rangle$. By Theorem 7.2.4 we immediately see that the first two graded pieces are

$$[M^0_{0,c}(\texttt{stand})/\left\langle v_\pm, \sigma_3 \otimes b_\pm \right\rangle] = [\texttt{stand}]$$

and

$$[M^1_{0,c}(\texttt{stand})/\left\langle v_\pm, \sigma_3 \otimes b_\pm \right\rangle] = ([\texttt{triv} + [\texttt{sign}] + [\texttt{stand}]) - [\texttt{stand}] = [\texttt{triv} + [\texttt{sign}].$$

Let us consider degree 2. By Theorem 7.2.4, a basis for $M^2_{0,c}(\texttt{stand})$ is

$$\begin{aligned}
\texttt{stand} :& \sigma_2 \otimes b_+ \\
& \sigma_2 \otimes b_- \\
\texttt{stand} :& -(-b_+^2 + 3b_-^2) \otimes b_+ + 3 \cdot (2b_+b_-) \otimes b_- \\
& (-b_+^2 + 3b_-^2) \otimes b_- + (2b_+b_-) \otimes b_+ \\
\texttt{triv} :& (-b_+^2 + 3b_-^2) \otimes b_+ + 3 \cdot (2b_+b_-) \otimes b_- \\
\texttt{sign} :& (-b_+^2 + 3b_-^2) \otimes b_- - (2b_+b_-) \otimes b_+.
\end{aligned}$$

Also using Theorem 7.2.4, a basis for $\langle v_\pm \rangle \cap M^2_{0,c}(\texttt{stand})$ is

$$\begin{aligned}
\texttt{triv} :& b_+v_+ + 3b_-v_- = (-b_+^2 + 3b_-^2) \otimes b_+ + 3 \cdot (2b_+b_-) \otimes b_- \\
\texttt{sign} :& b_+v_- - b_-v_+ = -((-b_+^2 + 3b_-^2) \otimes b_- - (2b_+b_-) \otimes b_+) \\
\texttt{stand} :& -b_+v_+ + 3b_-v_- = \sigma_2 \otimes b_+ \\
& b_+v_- + b_-v_+ = \sigma_2 \otimes b_-.
\end{aligned}$$

Note this shows that $\sigma_2 \otimes b_\pm$ are in $\langle v_\pm \rangle$, which explains why they do not show up in the statement of the lemma even though they are singular for every c. Taken together, this means that

$$[M^2_{0,c}(\texttt{stand})/\left\langle v_\pm, \sigma_3 \otimes b_\pm \right\rangle] = [\texttt{stand}].$$

Let us consider degree 3. By Theorem 7.2.4, a basis for $M^3_{0,c}(\texttt{stand})$ is

$$\begin{aligned}
\texttt{stand} :& \sigma_3 \otimes b_+ \\
& \sigma_3 \otimes b_- \\
\texttt{stand} :& q \otimes b_- \\
& q \otimes b_+ \\
\texttt{stand} :& \sigma_2(-b_+ \otimes b_+ + 3b_- \otimes b_-) \\
& \sigma_2(b_+ \otimes b_- + b_- \otimes b_+) \\
\texttt{triv} :& \sigma_2(b_+ \otimes b_+ + 3b_- \otimes b_-)
\end{aligned}$$

$$\texttt{sign} : \sigma_2(b_+\otimes b_- - b_-\otimes b_+).$$

As our considerations in degree 2 show, $\sigma_i\otimes b_\pm \in \langle v_\pm, \sigma_3\otimes b_\pm\rangle$, so all the above vectors are in $\langle v_\pm, \sigma_3\otimes b_\pm\rangle$ except maybe $q\otimes b_\pm$. However, we note that

$$-(-b_+^2+3b_-^2)v_+ + 3\cdot(2b_+b_-)v_- = 108\sigma_3\otimes b_+ + 18q\otimes b_-,$$

so

$$q\otimes b_- \in \langle v_\pm, \sigma_3\otimes b_\pm\rangle$$

and so is $q\otimes b_+$. This shows that

$$[M_{0,c}^3(\texttt{stand})/\langle v_\pm, \sigma_3\otimes b_\pm\rangle] = 0,$$

and consequently so is $[M_{0,c}^k(\texttt{stand})/\langle v_\pm, \sigma_3\otimes b_\pm\rangle]$ for all $k\geq 3$.

This shows that the module $M_{0,c}(\texttt{stand})/\langle v_\pm, \sigma_3\otimes b_\pm\rangle$ indeed has the character stated in the theorem. The irreducible module $L_{0,c}(\texttt{stand})$ is some quotient of $M_{0,c}(\texttt{stand})/\langle v_\pm, \sigma_3\otimes b_\pm\rangle$, but [DeSa14] Proposition 4.1 tells us they have the same character and are therefore equal. □

11.2 Singular vectors in $M_{1,c}(\texttt{stand})$ characteristic $p>3$ for generic c

The remaining task when $t=1$, $\tau=\texttt{stand}$ and c is generic is to explicitly describe the singular vectors in $M_{1,c}(\texttt{stand})$. We will do this by computing Dunkl operators using the basis from Theorem 7.2.4.

By Proposition 3.4 of [BaCh13a], for generic c singular vectors only appear in degrees divisible by p. By Lemma 10.0.2, any singular vectors in degrees divisible by p are in the isotypic component of stand. As stand is irreducible, it is enough to look for singular vectors in the 1-dimensional subspace of stand which restricts to the trivial representation of S_2 (in other words, the image of b_+ under any isomorphism from stand). By Theorem 7.2.4, a basis of this part of stand in degree kp is the union of the bases:

$$\begin{aligned}\{\sigma_2^a\sigma_3^b\otimes b_+ \mid 2a+3b=kp\},\\ \{\sigma_2^a\sigma_3^b\cdot(-b_+\otimes b_+ + 3b_-\otimes b_-) \mid 2a+3b=kp-1\},\\ \{\sigma_2^a\sigma_3^b\cdot\big(-(-b_+^2+3b_-^2)\otimes b_+ + 3(2b_+b_-)\otimes b_-\big) \mid 2a+3b=kp-2\},\\ \{\sigma_2^a\sigma_3^b q\otimes b_- \mid 2a+3b=kp-3\}.\end{aligned}$$

By Lemma 10.0.4, looking for singular vectors in degree kp means looking for a linear combination of the above vectors which is in the kernel of D_{y_1}. We first calculate the values of the Dunkl operator D_{y_1} on the vectors listed above.

Lemma 11.2.1. *1. Let $a,b,k\in\mathbb{N}_0$, $2a+3b=kp$. Then*

$$D_{y_1}(\sigma_2^a\sigma_3^b\otimes b_+)=\left(\frac{-a}{6}\right)\sigma_2^{a-1}\sigma_3^b\,(b_+\otimes b_++3b_-\otimes b_+)$$
$$+\left(\frac{b}{36}\right)\sigma_2^a\sigma_3^{b-1}\left((-b_+^2+3b_-^2)\otimes b_++3(2b_+b_-)\otimes b_+\right).$$

2. Let $a,b,k\in\mathbb{N}_0$, $2a+3b=kp-1$. Then

$$D_{y_1}(-\sigma_2^a\sigma_3^b b_+\otimes b_++3\sigma_2^a\sigma_3^b b_-\otimes b_-)=$$
$$=\left(\frac{-1}{2}\right)\sigma_2^a\sigma_3^b\otimes(b_+-3b_-)$$
$$+\left(\frac{-b}{6}\right)\sigma_2^{a+1}\sigma_3^{b-1}\,(b_+\otimes b_+-3b_-\otimes b_++3b_+\otimes b_-+3b_-\otimes b_-)$$
$$+\left(\frac{-a}{12}\right)\sigma_2^{a-1}\sigma_3^b\left((-b_+^2+3b_-^2)\otimes b_+-3(2b_+b_-)\otimes b_+\right.$$
$$\left.+3(-b_+^2+3b_-^2)\otimes b_-+3(2b_+b_-)\otimes b_-\right)$$
$$+\left(\frac{-b}{2}\right)\sigma_2^a\sigma_3^{b-1}q\otimes(b_++b_-).$$

3. Let $a,b,k\in\mathbb{N}_0$, $2a+3b=kp-2$. Then

$$D_{y_1}(-\sigma_2^a\sigma_3^b(-b_+^2+3b_-^2)\otimes b_++3\sigma_2^a\sigma_3^b(2b_+b_-)\otimes b_-)=$$
$$=(-2b)\sigma_2^{a+2}\sigma_3^{b-1}\otimes(b_+-3b_-)$$
$$+(9a)\sigma_2^{a-1}\sigma_3^{b+1}\otimes(b_+-3b_-)$$
$$+(-a)\sigma_2^a\sigma_3^b\,(b_+\otimes b_+-3b_-\otimes b_++3b_+\otimes b_-+3b_+\otimes b_-)$$
$$+(18c)\sigma_2^a\sigma_3^b\,(b_+\otimes b_--b_-\otimes b_-)$$
$$+\left(\frac{b}{6}\right)\sigma_2^{a+1}\sigma_3^{b-1}\left((-b_+^2+3b_-^2)\otimes b_+-3(2b_+b_-)\otimes b_+\right.$$
$$\left.+3(-b_+^2+3b_-^2)\otimes b_-+3(2b_+b_-)\otimes b_-\right)$$
$$+(-3a)\sigma_2^{a-1}\sigma_3^b q\otimes(b_++b_-).$$

4. Let $a,b,k\in\mathbb{N}_0$, $2a+3b=kp-3$. Then

$$D_{y_1}(\sigma_2^a\sigma_3^b q\otimes b_-)=c\sigma_2^{a+1}\sigma_3^b\otimes(b_+-3b_-)$$
$$+\frac{b}{3}\sigma_2^{a+2}\sigma_3^{b-1}\,(b_+\otimes b_--b_-\otimes b_-)$$
$$-\frac{3a}{2}\sigma_2^{a-1}\sigma_3^{b+1}\,(b_+\otimes b_--b_-\otimes b_-)+$$

$$+\left(-\frac{c}{12}\right)\sigma_2^a\sigma_3^b\left((-b_+^2+3b_-^2)\otimes b_+-3(2b_+b_-)\otimes b_+\right.$$
$$\left.+3(-b_+^2+3b_-^2)\otimes b_-+3(2b_+b_-)\otimes b_-\right).$$

Proof. Direct computation using the auxiliary results in Chapter 10. We shall prove part (1), but the other parts are proved similarly.

Since $\sigma_2^a\sigma_3^b$ is S_3 invariant, we have

$$\left(\frac{1-s}{\alpha_s}\otimes s\right)(\sigma_2^a\sigma_3^b\otimes b_+)=0$$

for all $s\in S$ and all $a,b\in\mathbb{N}_0$. Hence

$$\begin{aligned}
D_{y_1}(\sigma_2^a\sigma_3^b\otimes b_+) &= \partial_{y_1}(\sigma_2^a\sigma_3^b)\otimes b_+\\
&=\left(\partial_{y_1}(\sigma_2)\cdot a\cdot\sigma_2^{a-1}\sigma_3^b+\partial_{y_1}(\sigma_3^b)\cdot b\cdot\sigma_2^a\sigma_3^{b-1}\right)\otimes b_+\\
&=\left(\frac{-1}{6}b_++\frac{-1}{2}b_-\right)\cdot a\cdot\sigma_2^{a-1}\sigma_3^b\otimes b_+\\
&\quad+\left(\frac{1}{36}(-b_+^2+3b_3^2)+\frac{1}{12}(2b_+b_-)\right)\cdot b\cdot\sigma_2^a\sigma_3^{b-1}\otimes b_+\\
&=\left(\frac{-a}{6}\right)\sigma_2^{a-1}\sigma_3^b\left(b_+\otimes b_++3b_-\otimes b_+\right)\\
&\quad+\left(\frac{b}{36}\right)\sigma_2^a\sigma_3^{b-1}\left((-b_+^2+3b_-^2)\otimes b_++3(2b_+b_-)\otimes b_+\right).
\end{aligned}$$

□

Let us first look for singular vectors in degree p first (which will turn out to be enough). Parametrising the integers a,b which label the basis from the start of this section like in the proof of Lemma 7.2.1, we are looking for a vector in degree p of the form

$$\begin{aligned}
v_+ = &\sum_{0\le j\le\lfloor\frac{p-3}{6}\rfloor}\alpha_j\sigma_2^{\frac{p-3}{2}-3j}\sigma_3^{2j+1}\otimes b_+\\
&+\sum_{0\le j\le\lfloor\frac{p-1}{6}\rfloor}\beta_j\sigma_2^{\frac{p-1}{2}-3j}\sigma_3^{2j}\cdot(-b_+\otimes b_++3b_-\otimes b_-)\\
&+\sum_{0\le j\le\lfloor\frac{p-5}{6}\rfloor}\gamma_j\sigma_2^{\frac{p-5}{2}-3j}\sigma_3^{2j+1}\left(-(-b_+^2+3b_-^2)\otimes b_++3(2b_+b_-)\otimes b_-\right)\\
&+\sum_{0\le j\le\lfloor\frac{p-3}{6}\rfloor}\delta_j\sigma_2^{\frac{p-3}{2}-3j}\sigma_3^{2j}\cdot q\otimes b_-
\end{aligned}\tag{11.2.3}$$

for some $\alpha_j,\beta_j,\gamma_j,\delta_j\in\Bbbk[c]$, with the property that $D_{y_1}(v_+)=0$.

Lemma 11.2.4. *If a vector v_+ of the form* (11.2.3) *satisfies $D_{y_1}(v_+)=0$, then*

1. $\alpha_j = 0$ *for all* j*;*

2. *For all* $0 \le j \le \lfloor \frac{p-7}{6} \rfloor$

$$\gamma_j = \frac{2(j+1)}{3(6j+5)}\beta_{j+1}.$$

Note that if $p \equiv 2 \pmod 3$ then there is also a coefficient $\gamma_{\frac{p-5}{6}}$, which this lemma puts no conditions on. Otherwise this lemma determines all α_j and all γ_j in terms of β_j.

Proof. The right hand sides in each of the expressions in Lemma 11.2.1 are linearly independent, so we will be reading off their coefficients in the expansion of the equation $D_{y_1}(v_+) = 0$ using Lemma 11.2.1.

1. The coefficient of $(b_+ \otimes b_+ + 3b_- \otimes b_+)$ in the equation $D_{y_1}(v_+) = 0$ expanded as in Lemma 11.2.1 equals

$$\sum_{0\le j\le \lfloor \frac{p-3}{6}\rfloor} \alpha_j \left(-\frac{\frac{p-3}{2}-3j}{6}\right)\sigma_2^{\frac{p-3}{2}-3j-1}\sigma_3^{2j+1} = 0.$$

This tells us that $\alpha_j = 0$ for all $0 \le j \le \lfloor \frac{p-3}{6} \rfloor$, except maybe for j such that

$$\frac{p-3}{2} - 3j = 0, \quad 0 \le j \le \left\lfloor \frac{p-3}{6} \right\rfloor.$$

Keeping in mind that the equation $\frac{p-3}{2} - 3j = 0$ is in $\Bbbk$ but the inequality $0 \le j \le \lfloor \frac{p-3}{6} \rfloor$ is in $\mathbb{N}_0$, this is equivalent to

$$j = \frac{p-3}{6} \in \mathbb{N}_0.$$

However, this means that $6 \mid (p-3)$ so $3 \mid p$, which is impossible as p is a prime and $p > 3$. In conclusion, no such j exists, so $\alpha_j = 0$ for all j.

2. The coefficient of $q \otimes (b_+ + b_-)$ in the equation $D_{y_1}(v_+) = 0$ expanded as in Lemma 11.2.1 equals

$$\sum_{0\le j\le \lfloor \frac{p-1}{6}\rfloor} \beta_j \left(\frac{-2j}{2}\right)\sigma_2^{\frac{p-1}{2}-3j}\sigma_3^{2j-1} + \sum_{0\le j\le \lfloor \frac{p-5}{6}\rfloor} \gamma_j(-3)\left(\frac{p-5}{2}-3j\right)\sigma_2^{\frac{p-5}{2}-3j-1}\sigma_3^{2j+1} = 0.$$

Noting that the first summand is zero for $j = 0$, we perform the change of summation variable $j \to j+1$ to rewrite the sum as

$$-\sum_{0\le j\le \lfloor \frac{p-7}{6}\rfloor} \beta_{j+1}\,(j+1)\,\sigma_2^{\frac{p-7}{2}-3j}\sigma_3^{2j+1} + \left(\frac{-3}{2}\right)\sum_{0\le j\le \lfloor \frac{p-5}{6}\rfloor} \gamma_j\,(p-5-6j)\,\sigma_2^{\frac{p-7}{2}-3j}\sigma_3^{2j+1} = 0.$$

The boundaries of these two sums are different precisely when there exists an integer j with $\frac{p-7}{6} < j \le \frac{p-5}{6}$, which is equivalent to $6j = p-6$ or $6j = p-5$. If $6j = p-6$

then $6 \mid p$ which is impossible because p is prime. If $6j=p-5$, which happens exactly when $p\equiv 2 \pmod 3$, then the summand with coefficient $\gamma_{\frac{p-5}{6}}$ is zero, so for any p the above equation becomes

$$\sum_{0\le j\le \lfloor \frac{p-7}{6}\rfloor} \left(-\beta_{j+1}(j+1)+\left(\frac{-3}{2}\right)\gamma_j(p-5-6j)\right)\sigma_2^{\frac{p-7}{2}-3j}\sigma_3^{2j+1}=0.$$

This means that for all $0\le j\le \lfloor \frac{p-7}{6}\rfloor$

$$\gamma_j=\frac{2(j+1)}{3(6j+5)}\beta_{j+1}.$$

□

Using Lemma 11.2.4, any vector v_+ of the form (11.2.3) which satisfies $D_{y_1}(v_+)=0$ is of the form

$$\begin{aligned} v_+ = &\sum_{0\le j\le \lfloor \frac{p-1}{6}\rfloor} \beta_j\sigma_2^{\frac{p-1}{2}-3j}\sigma_3^{2j}\cdot(-b_+\otimes b_+ + 3b_-\otimes b_-) \\ &+\sum_{0\le j\le \lfloor \frac{p-7}{6}\rfloor} \frac{2(j+1)}{3(6j+5)}\beta_{j+1}\sigma_2^{\frac{p-5}{2}-3j}\sigma_3^{2j+1}\left(-(-b_+^2+3b_-^2)\otimes b_+ + 3(2b_+b_-)\otimes b_-\right) \\ &+\delta_{p\equiv 2\,(\mathrm{mod}\,3)}\cdot\gamma_{\frac{p-5}{6}}\sigma_3^{\frac{p-2}{3}}\left(-(-b_+^2+3b_-^2)\otimes b_+ + 3(2b_+b_-)\otimes b_-\right) \\ &+\sum_{0\le j\le \lfloor \frac{p-3}{6}\rfloor} \delta_j\sigma_2^{\frac{p-3}{2}-3j}\sigma_3^{2j}\cdot q\otimes b_- \end{aligned} \tag{11.2.5}$$

where $\delta_{p\equiv 2\,(\mathrm{mod}\,3)}$ denotes the delta function

$$\delta_{p\equiv 2\,(\mathrm{mod}\,3)}=\begin{cases}1 & \text{if } p\equiv 2 \pmod 3\\ 0 & \text{otherwise.}\end{cases}$$

Next, we want to establish the conditions on β_j, δ_j, and (where needed) $\gamma_{\frac{p-5}{6}}$.

Lemma 11.2.6. *For a vector v_+ of the form (11.2.5) the condition $D_{y_1}(v_+)=0$ is equivalent to the system of equations:*

1. *If $p\equiv 1 \pmod 3$:*

   For all $0\le j\le \frac{p-1}{6}-1$

$$-\frac{6j+1}{2}\beta_j-\frac{4(j+1)(2j+1)}{3(6j+5)}\beta_{j+1}+c\delta_j=0 \tag{I}$$

For all $0 \leq j \leq \frac{p-1}{6} - 2$

$$\frac{12(j+1)c}{(6j+5)}\beta_{j+1} + \frac{2(j+1)}{3}\delta_{j+1} + \frac{3(6j+3)}{4}\delta_j = 0, \tag{II}$$

$$c \cdot \beta_{\frac{p-1}{6}} - 3\delta_{\frac{p-1}{6}-1} = 0. \tag{II'}$$

2. *If* $p \equiv 2 \pmod 3$*:*

   For all $0 \leq j \leq \frac{p-5}{6} - 1$

$$-\frac{6j+1}{2}\beta_j - \frac{4(j+1)(2j+1)}{3(6j+5)}\beta_{j+1} + c\delta_j = 0 \tag{I}$$

$$2\beta_{\frac{p-5}{6}} + \frac{4}{3}\gamma_{\frac{p-5}{6}} + c\delta_{\frac{p-5}{6}} = 0 \tag{I'}$$

   For all $0 \leq j \leq \frac{p-5}{6} - 1$

$$\frac{12(j+1)c}{(6j+5)}\beta_{j+1} + \frac{2(j+1)}{3}\delta_{j+1} + \frac{3(6j+3)}{4}\delta_j = 0 \tag{II}$$

$$\gamma_{\frac{p-5}{6}} \cdot 18c + \frac{-3}{2}\delta_{\frac{p-5}{6}} = 0 \tag{II'}$$

Proof. We look at all the right hand sides of the values of Dunkl operators on basis vectors from Lemma 11.2.1, and setting their coefficients in $D_{y_1}(v_+)$ to be 0 get the following equations:

(I) From the coefficient of $\otimes(b_+ - 3b_-)$:

$$\begin{aligned}
0 = & \sum_{0\leq j\leq\lfloor\frac{p-1}{6}\rfloor} \left(\frac{-1}{2}\right)\beta_j\sigma_2^{\frac{p-1}{2}-3j}\sigma_3^{2j} \\
&+ \sum_{0\leq j\leq\lfloor\frac{p-7}{6}\rfloor} \frac{2(j+1)}{3(6j+5)}\beta_{j+1}\cdot(-2)(2j+1)\sigma_2^{\frac{p-5}{2}-3j+2}\sigma_3^{2j} \\
&+ \sum_{0\leq j\leq\lfloor\frac{p-7}{6}\rfloor} \frac{2(j+1)}{3(6j+5)}\beta_{j+1}\cdot 9\left(\frac{p-5}{2} - 3j\right)\sigma_2^{\frac{p-5}{2}-3j-1}\sigma_3^{2j+2} \\
&+ \delta_{p\equiv 2 \pmod 3}\cdot\gamma_{\frac{p-5}{6}}(-2)\left(\frac{p-2}{3}\right)\sigma_2^2\sigma_3^{\frac{p-2}{3}-1} \\
&+ \sum_{0\leq j\leq\lfloor\frac{p-3}{6}\rfloor} \delta_j\cdot c\sigma_2^{\frac{p-3}{2}-3j+1}\sigma_3^{2j}
\end{aligned}$$

$$= \sum_{0\le j\le\lfloor\frac{p-1}{6}\rfloor}\left(\frac{-1}{2}\right)\beta_j\sigma_2^{\frac{p-1}{2}-3j}\sigma_3^{2j} + \sum_{0\le j\le\lfloor\frac{p-7}{6}\rfloor}\frac{-4(j+1)(2j+1)}{3(6j+5)}\beta_{j+1}\sigma_2^{\frac{p-1}{2}-3j}\sigma_3^{2j}$$
$$+ \sum_{0\le j\le\lfloor\frac{p-1}{6}\rfloor}(-3j)\beta_j\sigma_2^{\frac{p-1}{2}-3j}\sigma_3^{2j} + \delta_{p\equiv 2\,(\mathrm{mod}\,3)}\cdot\gamma_{\frac{p-5}{6}}\frac{(-2)(p-2)}{3}\sigma_2^2\sigma_3^{2\cdot\frac{p-5}{6}}$$
$$+ \sum_{0\le j\le\lfloor\frac{p-3}{6}\rfloor}\delta_j\cdot c\sigma_2^{\frac{p-1}{2}-3j}\sigma_3^{2j}$$

We now distinguish two cases:

1. If $p \equiv 1 \pmod 3$, then $\lfloor\frac{p-3}{6}\rfloor = \frac{p-7}{6}$, $\delta_{p\equiv 2\,(\mathrm{mod}\,3)} = 0$. Reading the coefficient of $\sigma_2^{\frac{p-1}{2}-3j}\sigma_3^{2j}$ we get that the above equation is equivalent to requiring for all $0 \le j \le \frac{p-7}{6}$ that
$$-\frac{6j+1}{2}\beta_j + \frac{-4(j+1)(2j+1)}{3(6j+5)}\beta_{j+1} + c\delta_j = 0.$$
which gives equation (I) from the Lemma. For $j = \frac{p-1}{6}$ the coefficient of $\sigma_2^{\frac{p-1}{2}-3j}\sigma_3^{2j}$ gives $0 = 0$, which is always satisfied.

2. If $p \equiv 2 \pmod 3$, then $\lfloor\frac{p-1}{6}\rfloor = \lfloor\frac{p-3}{6}\rfloor = \frac{p-5}{6}$, $\lfloor\frac{p-7}{6}\rfloor = \frac{p-5}{6} - 1$ and $\delta_{p\equiv 2\,(\mathrm{mod}\,3)} = 1$. Reading the coefficient of $\sigma_2^{\frac{p-1}{2}-3j}\sigma_3^{2j}$ we get that the above equation is equivalent to requiring for all $0 \le j \le \frac{p-5}{6} - 1$ that
$$\frac{-(6j+1)}{2}\beta_j + \frac{-4(j+1)(2j+1)}{3(6j+5)}\beta_{j+1} + c\delta_j = 0.$$
For $j = \frac{p-5}{6}$ the coefficient of $\sigma_2^2\sigma_3^{2j}$ gives one additional equation
$$\frac{-p+4}{2}\beta_j + \frac{-2}{3}(p-2)\gamma_{\frac{p-5}{6}} + c\delta_{\frac{p-5}{6}} = 0.$$
Using that $p = 0 \in \Bbbk$ we get exactly the equations (I) and (I') from the Lemma.

(II) From the coefficient of $(b_+ \otimes b_- - b_- \otimes b_-)$:

$$0 = \sum_{0\le j\le\lfloor\frac{p-7}{6}\rfloor}\frac{2(j+1)}{3(6j+5)}\beta_{j+1}\cdot 18c\cdot\sigma_2^{\frac{p-5}{2}-3j}\sigma_3^{2j+1} + \delta_{p\equiv 2\bmod 3}\cdot\gamma_{\frac{p-5}{6}}\cdot 18c\cdot\sigma_3^{2\frac{p-5}{6}+1}$$
$$+ \sum_{0\le j\le\lfloor\frac{p-9}{6}\rfloor}\delta_{j+1}\frac{2(j+1)}{3}\sigma_2^{\frac{p-5}{2}-3j}\sigma_3^{2j+1} + \sum_{0\le j\le\lfloor\frac{p-3}{6}\rfloor}\delta_j\frac{-3(\frac{p-3}{2}-3j)}{2}\sigma_2^{\frac{p-5}{2}-3j}\sigma_3^{2j+1}.$$

We now distinguish two cases:

1. If $p \equiv 1 \pmod 3$, then for all $0 \le j \le \frac{p-1}{6} - 2$

$$\frac{12(j+1)c}{(6j+5)}\beta_{j+1} + \frac{2(j+1)}{3}\delta_{j+1} + \frac{3(6j+3)}{4}\delta_j = 0.$$

For $j = \frac{p-1}{6} - 1$ we get an additional equation

$$c \cdot \beta_{\frac{p-1}{6}} - 3\delta_{\frac{p-1}{6}-1} = 0.$$

which are exactly equations (II) and (II') from the Lemma.

2. If $p \equiv 2 \pmod 3$, then for all $0 \le j \le \frac{p-5}{6} - 1$

$$\frac{12(j+1)c}{(6j+5)}\beta_{j+1} + \frac{2(j+1)}{3}\delta_{j+1} + \frac{3(6j+3)}{4}\delta_j = 0.$$

For $j = \frac{p-5}{6}$ we get an additional equation

$$\gamma_{\frac{p-5}{6}} \cdot 18c + \frac{-3}{2}\delta_{\frac{p-5}{6}} = 0.$$

which are exactly equations (II) and (II') from the Lemma.

(III) The coefficient of $b_+ \otimes b_+ - 3b_- \otimes b_+ + 3b_+ \otimes b_- + 3b_- \otimes b_-$ in $D_{y_1}(v_+) = 0$ is automatically satisfied when v_+ is of the form (11.2.5).

(IV) From the coefficient of $(-b_+^2 + 3b_-^2) \otimes b_+ - 3(2b_+b_-) \otimes b_+ + 3(-b_+^2 + 3b_-^2) \otimes b_- + 3(2b_+b_-) \otimes b_-$ we get conditions proportional to (I), (I').

□

Lemma 11.2.7. *For every $p > 3$, the system of equations from the statement of Lemma 11.2.6 has a unique solution up to overall scaling.*

Proof. 1. If $p \equiv 1 \pmod 3$ and we write $p = 6k+1$, then the unknowns are $\beta_0, \beta_1, \dots, \beta_k$ and $\delta_0, \delta_1, \dots, \delta_{k-1}$. Ordering them as

$$\beta_k, \delta_{k-1}, \beta_{k-1}, \dots, \delta_0, \beta_0,$$

we can treat equations (I),(II),(II') as recursions that let us calculate each unknown from the previous two. More precisely, choose β_k to be arbitrary, then use (II') to calculate

$$\delta_{k-1} = \frac{c}{3} \cdot \beta_k.$$

After that, alternating (I) and (II) lets us calculate the remaining unknowns recursively:

$$\delta_j = \frac{-4}{3(6j+3)}\left(\frac{12(j+1)c}{(6j+5)}\beta_{j+1} + \frac{2(j+1)}{3}\delta_{j+1}\right),$$

$$\beta_j = \frac{2}{6j+1}\left(\frac{-4(j+1)(2j+1)}{3(6j+5)}\beta_{j+1} + c\delta_j\right).$$

2. If $p \equiv 2 \pmod 3$ and we write $p = 6k+5$, then the unknowns are $\beta_0, \beta_1, \ldots, \beta_k$, γ_k, and $\delta_0, \delta_1, \ldots, \delta_k$. We allow γ_k to be arbitrary, then use (II') to calculate

$$\delta_k = 12c \cdot \gamma_k$$

and (I') to calculate

$$\beta_k = -\frac{2}{3}\gamma_k - \frac{c}{2}\delta_k.$$

After that, alternating (I) and (II) lets us calculate the remaining unknowns recursively, for $0 \le j \le k-1$ in decreasing order of j, as

$$\delta_j = \frac{-16(j+1)c}{3(6j+5)(2j+1)}\beta_{j+1} - \frac{8(j+1)}{27(2j+1)}\delta_{j+1}$$

$$\beta_j = \frac{-8(j+1)(2j+1)}{3(5+6j)(6j+1)}\beta_{j+1} + \frac{2c}{6j+1}\delta_j.$$

□

Corollary 11.2.8. *For every $p > 3$, $t = 1$, and $c \notin \mathbb{F}_p$, there is a 2-dimensional space of singular vectors in $M^p_{1,c}(\texttt{stand})$. These singular vectors v_+, v_- are unique up to overall scaling, and can be determined by Lemma 11.2.7 by choosing a nonzero constant as the first term and applying relations recursively.*

Example 11.2.9. When $p = 5$, a solution of this system is

$$\gamma_0 = 1, \delta_0 = 2c, \beta_0 = 1 - c^2,$$

leading to

$$\begin{aligned} v_+ &= (1-c^2)\sigma_2^2 \cdot (-b_+ \otimes b_+ + 3b_- \otimes b_-) \\ &\quad + \sigma_3\left(-(-b_+^2 + 3b_-^2) \otimes b_+ + 3(2b_+b_-) \otimes b_-\right) + 2c\sigma_2 \cdot q \otimes b_-, \\ v_- &= (1-c^2)\sigma_2^2 \cdot (b_+ \otimes b_- + b_- \otimes b_+) \\ &\quad + \sigma_3\left((-b_+^2 + 3b_-^2) \otimes b_- + (2b_+b_-) \otimes b_+\right) + c\sigma_2 \cdot q \otimes b_+. \end{aligned}$$

Example 11.2.10. When $p = 7$, a solution of this system is

$$\beta_1 = 2,\ \delta_0 = 3c,\ \beta_0 = 6c^2 - 2$$

leading to

$$v_+ = \left((6c^2-2)\sigma_2^3 + 2\sigma_3^2\right) \cdot (-b_+ \otimes b_+ + 3b_- \otimes b_-)$$

$$
\begin{aligned}
&+ 4\sigma_2^{\frac{p-5}{2}}\sigma_3\left(-(-b_+^2+3b_-^2)\otimes b_+ + 3(2b_+b_-)\otimes b_-\right)\\
&+ 3c\sigma_2^2\cdot q\otimes b_-,\\
v_- = &\left((6c^2-2)\sigma_2^3+2\sigma_3^2\right)\cdot(b_+\otimes b_- + b_-\otimes b_+)\\
&+ 4\sigma_2^{\frac{p-5}{2}}\sigma_3\left((-b_+^2+3b_-^2)\otimes b_- + (2b_+b_-)\otimes b_+\right)\\
&- c\sigma_2^2\cdot q\otimes b_-.
\end{aligned}
$$

Example 11.2.11. When $p=11$, a solution of this system is

$$\gamma_1=3,\ \delta_1=3c,\ \beta_1=4c^2+9,\ \delta_0=c(6c^2+1),\ \beta_0=c^4+5c^2+4$$

leading to

$$
\begin{aligned}
v_+ = &\left((c^4+5c^2+4)\sigma_2^5+(4c^2+9)\sigma_2^2\sigma_3^2\right)\cdot(-b_+\otimes b_+ + 3b_-\otimes b_-)+\\
&+ (2c^2-1)\sigma_2^3\sigma_3\left(-(-b_+^2+3b_-^2)\otimes b_+ + 3(2b_+b_-)\otimes b_-\right)\\
&+ c(6c^2+1)\sigma_2^4\cdot q\otimes b_- + 3c\sigma_2\sigma_3^2\cdot q\otimes b_-,\\
v_- = &\left((c^4+5c^2+4)\sigma_2^5+(4c^2+9)\sigma_2^2\sigma_3^2\right)\cdot(b_+\otimes b_- + b_-\otimes b_+)\\
&+ (2c^2-1)\sigma_2^3\sigma_3\left((-b_+^2+3b_-^2)\otimes b_- + (2b_+b_-)\otimes b_+\right)\\
&- 4c(6c^2+1)\sigma_2^4\cdot q\otimes b_+ - c\sigma_2\sigma_3^2\cdot q\otimes b_+.
\end{aligned}
$$

Example 11.2.12. When $p=13$, a solution of this system is

$$\beta_2=3,\delta_1=c,\beta_1=9+4c^2,\delta_0=c(7c^2-2),\beta_0=c^4+6c^2+3,$$

leading to

$$
\begin{aligned}
v_+ = &\left((c^4+6c^2+3)\sigma_2^6+(9+4c^2)\sigma_2^3\sigma_3^2+3\sigma_3^4\right)\cdot(-b_+\otimes b_+ + 3b_-\otimes b_-)\\
&+ \left((9+4c^2)\sigma_2^4\sigma_3-2\sigma_2\sigma_3^3\right)\cdot\left(-(-b_+^2+3b_-^2)\otimes b_+ + 3(2b_+b_-)\otimes b_-\right)\\
&+ \left(c(7c^2-2)\sigma_2^5+c\sigma_2^2\sigma_3^2\right)\cdot q\otimes b_-,\\
v_- = &\left((c^4+6c^2+3)\sigma_2^6+(9+4c^2)\sigma_2^3\sigma_3^2+3\sigma_3^4\right)\cdot(b_+\otimes b_- + b_-\otimes b_+)\\
&+ \left((9+4c^2)\sigma_2^4\sigma_3-2\sigma_2\sigma_3^3\right)\cdot\left((-b_+^2+3b_-^2)\otimes b_- + (2b_+b_-)\otimes b_+\right)\\
&+ \frac{-1}{3}\left(c(7c^2-2)\sigma_2^5+c\sigma_2^2\sigma_3^2\right)\cdot q\otimes b_+.
\end{aligned}
$$

Calculating a determinant

The first part of this section showed that for every $p>3$, at $t=1$, and for all c, the space of singular vectors in the `stand` isotypic component of $M_{1,c}^p(\texttt{stand})$ is 2-dimensional and spanned by v_+, v_- from Corollary 11.2.8. They satisfy

$$D_{y_1}(v_+) = D_{y_1}(v_-) = 0, \quad v_- = \frac{2}{3}\left(s_2 + \frac{1}{2}\mathrm{id}\right)v_+,$$

and their span is isomorphic to `stand` via $b_\pm \mapsto v_\pm$. We would now like to understand the submodule generated by these two vectors. We do that in the remainder of this section, using an argument similar to that in the proof of Lemma 8.3.3.

Let us name the components of v_+, v_- as

$$\begin{aligned} v_+ &= a_{++} \otimes b_+ + a_{+-} \otimes b_- \\ v_- &= a_{-+} \otimes b_+ + a_{--} \otimes b_-. \end{aligned}$$

To understand the submodule generated by v_+, v_-, we must analyse the determinant

$$\Delta = a_{++}a_{--} - a_{+-}a_{-+} \in S^{2p}(\mathfrak{h}^*).$$

Theorem 11.2.13. *1. Δ is an invariant in degree $2p$.*

2. Δ is in the kernel of ∂_{y_1}.

3. Δ is of the form $\Delta = f(c)\cdot\sigma_2^p$, where $f(c) \in \Bbbk[c]$ is a polynomial in c.

Proof. This proof does not use the explicit form of v_+ from the previous subsection; everything follows from the few equations displayed above this theorem.

1. The facts that

$$s_1.v_+ = v_+ \quad s_1.v_- = -v_-$$

imply that

$$s_1.a_{++} = a_{++}, \quad s_1.a_{+-} = -a_{+-}, \quad s_1.a_{-+} = -a_{-+}, \quad s_1.a_{--} = a_{--}.$$

From this immediately follows that

$$\begin{aligned} s_1.\Delta &= s_1.(a_{++}a_{--} - a_{+-}a_{-+}) \\ &= (s_1.a_{++})(s_1.a_{--}) - (s_1.a_{+-})(s_1.a_{-+}) \\ &= a_{++}a_{--} - (-a_{+-})(-a_{-+}) \\ &= a_{++}a_{--} - a_{+-}a_{-+} \\ &= \Delta. \end{aligned}$$

To prove the s_2 invariance of Δ, let us expand the formula

$$v_- = \frac{2}{3}\left(s_2 + \frac{1}{2}\mathrm{id}\right)v_+$$

as

$$\begin{aligned}
&a_{-+}\otimes b_+ + a_{--}\otimes b_- = \\
&\quad = \frac{2}{3}\left(s_2 + \frac{1}{2}\mathrm{id}\right)(a_{++}\otimes b_+ + a_{+-}\otimes b_-) \\
&\quad = \frac{2}{3}\left((s_2.a_{++})\otimes\frac{-b_+ + 3b_-}{2} + (s_2.a_{+-})\otimes\frac{b_+ + b_-}{2} + \frac{1}{2}a_{++}\otimes b_+ + \frac{1}{2}a_{+-}\otimes b_-\right)
\end{aligned}$$

which leads to:

$$\begin{aligned}
a_{-+} &= \frac{1}{3}\Big(-(s_2.a_{++}) + (s_2.a_{+-}) + a_{++}\Big) \\
a_{--} &= \frac{1}{3}\Big(3(s_2.a_{++}) + (s_2.a_{+-}) + a_{+-}\Big)
\end{aligned} \tag{11.2.14}$$

Using this we can rewrite Δ as:

$$\begin{aligned}
\Delta &= a_{++}a_{--} - a_{+-}a_{-+} \\
&= a_{++}\cdot\frac{1}{3}\Big(3(s_2.a_{++}) + (s_2.a_{+-}) + a_{+-}\Big) - a_{+-}\cdot\frac{1}{3}\Big(-(s_2.a_{++}) + (s_2.a_{+-}) + a_{++}\Big) \\
&= a_{++}(s_2.a_{++}) + \frac{1}{3}a_{++}(s_2.a_{+-}) + \frac{1}{3}a_{+-}(s_2.a_{++}) - \frac{1}{3}a_{+-}(s_2.a_{+-}).
\end{aligned}$$

This form makes it clear that Δ is an s_2 invariant, as the action of s_2 preserves the first and the fourth summand and swaps the second and third. So, Δ is invariant under the generators s_1 and s_2 of S_3, so it is an S_3 invariant.

2. We can use the formulas of (11.2.14) to deduce:

$$\begin{aligned}
(s_2.a_{++}) &= \frac{1}{4}\Big(a_{++} - a_{+-} - 3a_{-+} + 3a_{--}\Big) \\
(s_2.a_{+-}) &= \frac{1}{4}\Big(-3a_{++} - a_{+-} + 9a_{-+} + 3a_{--}\Big).
\end{aligned} \tag{11.2.15}$$

Similarly, the formula for the action of s_2 on v_- gives us:

$$\begin{aligned}
(s_2.a_{-+}) &= \frac{1}{4}\Big(-a_{++} + a_{+-} - a_{-+} + a_{--}\Big) \\
(s_2.a_{--}) &= \frac{1}{4}\Big(3a_{++} + a_{+-} + 3a_{-+} + a_{--}\Big).
\end{aligned} \tag{11.2.16}$$

From this we can also easily deduce formulas for $(13).a_{++}$ etc.

As we know the action of the group on all $a_{\pm\pm}$, we can expand $D_{y_1}(v_\pm)=0$ to get

$$\begin{aligned}
\partial_{y_1}(a_{++}) &= \frac{-c}{2(x_1-x_3)}\Big(a_{+-}-3a_{-+}\Big) \\
\partial_{y_1}(a_{+-}) &= \frac{-2c}{x_1-x_2}a_{+-}+\frac{c}{2(x_1-x_3)}\Big(-3a_{++}+2a_{+-}+3a_{--}\Big) \\
\partial_{y_1}(a_{-+}) &= \frac{2c}{x_1-x_2}a_{-+}-\frac{c}{2(x_1-x_3)}\Big(-a_{++}+2a_{-+}+a_{--}\Big) \\
\partial_{y_1}(a_{--}) &= \frac{c}{2(x_1-x_3)}\Big(a_{+-}-3a_{-+}\Big).
\end{aligned} \tag{11.2.17}$$

Using (11.2.17) we calculate

$$\begin{aligned}
\partial_{y_1}(\mathit{\Delta}) &= \partial_{y_1}(a_{++})a_{--}+\partial_{y_1}(a_{--})a_{++}-\partial_{y_1}(a_{+-})a_{-+}-\partial_{y_1}(a_{-+})a_{+-} \\
&= 0.
\end{aligned}$$

3. By (1) $\mathit{\Delta}$ is an invariant of degree $2p$, and by (2) it is in the kernel by ∂_{y_1}. As it is symmetric, it follows that it is also in the kernel by ∂_{y_2} and ∂_{y_3}, so $\mathit{\Delta}$ is a p-th power of a polynomial. The only symmetric polynomials in degree $2p$ which are also p-th powers are scalar multiples of σ_2^p.

□

In order to analyse the module $M_{1,c}(\texttt{stand})/\langle v_+,v_-\rangle$, we will need to show that v_+,v_- are (for generic c and some special c) as independent as they can possibly be, and this will be done by showing that $\mathit{\Delta}$ is nonzero (for generic c and some special c). This will include analysing the polynomial $f(c)\in\Bbbk[c]$ from Theorem 11.2.13 and its zeroes.

Lemma 11.2.18. *Let $p>3$, $t=1$, and c be generic. Let $f(c)\in\Bbbk[c]$ be the polynomial from Theorem 11.2.13.*

1. *It satisfies $f(c)=\frac{4}{3}(3\beta_0-\delta_0)(3\beta_0+\delta_0)$;*
2. *All coefficients β_j are even and all δ_j are odd polynomials in $\Bbbk[c]$.*
3. *If $p=6k+1$ then $\deg\beta_j=2k-2j$, $\deg\delta_j=2k-2j-1$, and $\deg f=\frac{p-1}{3}$.*
4. *If $p=6k+5$ then $\deg\beta_j=2k-2j+2$, $\deg\delta_j=2k-2j+1$, and $\deg f=\frac{p+1}{3}$.*
5. *The polynomial $f(c)$ is not identically zero.*

Proof. For part (1), Theorem 11.2.13 (3) showed that $\mathit{\Delta}=a_{++}a_{--}-a_{+-}a_{-+}$ is a scalar multiple of σ_2^p, so we now calculate the scalar by considering only the parts of a_{++}, a_{--}, a_{+-}

and a_{-+} which will contribute to a power of σ_2. More specifically, any term divisible by σ_3 will not contribute to Δ. Disregarding all such terms, we can write:

$$\begin{aligned} a_{++} &= \beta_0\sigma_2^{\frac{p-1}{2}}\cdot(-b_+)+\sigma_3\cdot(\ldots)\\ a_{+-} &= \beta_0\sigma_2^{\frac{p-1}{2}}\cdot 3b_- + \delta_0\sigma_2^{\frac{p-3}{2}}\cdot q+\sigma_3\cdot(\ldots)\\ a_{-+} &= \beta_0\sigma_2^{\frac{p-1}{2}}\cdot b_- - \frac{1}{3}\delta_0\sigma_2^{\frac{p-3}{2}}\cdot q+\sigma_3\cdot(\ldots)\\ a_{--} &= \beta_0\sigma_2^{\frac{p-1}{2}}\cdot b_+ +\sigma_3\cdot(\ldots). \end{aligned}$$

From here we then compute:

$$\begin{aligned} \Delta &= a_{++}a_{--}-a_{+-}a_{-+} = \\ &= \beta_0\sigma_2^{\frac{p-1}{2}}(-b_+)\beta_0\sigma_2^{\frac{p-1}{2}}b_+ - \left(\beta_0\sigma_2^{\frac{p-1}{2}}3b_- + \delta_0\sigma_2^{\frac{p-3}{2}}q\right)\left(\beta_0\sigma_2^{\frac{p-1}{2}}b_- - \frac{1}{3}\delta_0\sigma_2^{\frac{p-3}{2}}q\right)+\sigma_3\cdot(\ldots)\\ &= \beta_0^2\sigma_2^{p-1}(-b_+^2-3b_-^2)+\frac{1}{3}\delta_0^2\sigma_2^{p-3}q^2+\sigma_3\cdot(\ldots)\\ &= 12\beta_0^2\sigma_2^p+\frac{1}{3}\delta_0^2\sigma_2^{p-3}(-27\sigma_3^2-4\sigma_2^3)+\sigma_3\cdot(\ldots)\\ &= \left(12\beta_0^2-\frac{4}{3}\delta_0^2\right)\sigma_2^p+\sigma_3\cdot(\ldots). \end{aligned}$$

Comparing this with Theorem 11.2.13 (3) shows that $\Delta = f(c)\sigma_2^p$ with

$$f(c) = 12\beta_0^2-\frac{4}{3}\delta_0^2 = \frac{4}{3}\left(3\beta_0-\delta_0\right)\left(3\beta_0+\delta_0\right).$$

Parts (2), (3), and (4) follow from the recursions in Lemma 11.2.7.
For part (5), it is enough to see that the polynomial $f(c)$ is nonzero at a specific point $c=0$. Then $\delta_j=0$ for all j and $f(0)=12\beta_0^2$, which is nonzero by the recursions in Lemma 11.2.7.

□

We will now use this determinant calculation to analyse the modules $M_{1,c}(\texttt{stand})/\langle v_+,v_-\rangle$, and $M_{1,c}(\texttt{stand})/\langle v_\pm,\sigma_3^p\otimes b_\pm\rangle$.

Lemma 11.2.19. *Let $p>3$, $t=1$, and c be generic.*

1. *The submodule $\langle v_+,v_-\rangle$ of $M_{1,c}(\texttt{stand})$ is isomorphic to $M_{1,c}(\texttt{stand})[-p]$.*

2. *The vectors $\sigma_2^p\otimes b_+,\sigma_2^p\otimes b_-$ are contained in the submodule $\langle v_+,v_-\rangle$ of $M_{1,c}(\texttt{stand})$.*

3. *The vectors $\sigma_3^p\otimes b_+$ and $\sigma_3^p\otimes b_-$ are not contained in the submodule $\langle v_+,v_-\rangle$ of $M_{1,c}(\texttt{stand})$.*

4. *The module* $M_{1,c}(\texttt{stand})/\langle v_+, v_-, \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle$ *has the Hilbert polynomial:*

$$2\frac{(1-z^p)(1-z^{3p})}{(1-z)^2}.$$

5. *The module* $M_{1,c}(\texttt{stand})/\langle v_+, v_-, \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle$ *is irreducible and thus equal to* $L_{1,c}(\texttt{stand})$.

Proof. 1. The S_3 map $\varphi : \texttt{stand} \to M_{1,c}^p(\texttt{stand})$ given by $\varphi(b_\pm) = v_\pm$ induces a map of $H_{1,c}(S_3)$ modules $\varphi : M_{1,c}(\texttt{stand})[-p] \to M_{1,c}(\texttt{stand})$. Its image is the submodule $\langle v_+, v_- \rangle$, so all we have to show is that its kernel is zero.

Assume that $A, B \in S(\mathfrak{h}^*)$ are such that $A \otimes b_+ + B \otimes b_-$ is in the kernel of φ. We can assume without loss of generality that A, B are homogeneous of the same degree. Then we have

$$\varphi(A \otimes b_+ + B \otimes b_-) = 0$$

which can be rewritten as

$$A \cdot v_+ + B \cdot v_- = 0$$

and again as the system in $S(\mathfrak{h}^*)$

$$A \cdot a_{++} + B \cdot a_{-+} = 0$$
$$A \cdot a_{+-} + B \cdot a_{--} = 0.$$

Considering this as a linear system of equations, we note that its determinant is, by Theorem 11.2.13 and Lemma 11.2.18, equal to $f(c)\sigma_2^p$, which is nonzero for generic c, and thus for generic c the only solution of this system is $A = B = 0$. This shows that the kernel of φ is 0, so it is injective and its image $\langle v_+, v_- \rangle$ is isomorphic to $M_{1,c}(\texttt{stand})[-p]$.

2. Checking if the vector $\sigma_2^p \otimes b_+$ is contained in the $H_{t,c}(S_3)$ submodule $\langle v_+, v_- \rangle$ is equivalent to trying to find solutions $A, B \in S(\mathfrak{h}^*)$ such that

$$A \otimes b_+ + B \otimes b_- = \sigma_2^p \otimes b_+.$$

This can be rewritten as the system

$$A \cdot a_{++} + B \cdot a_{-+} = \sigma_2^p$$
$$A \cdot a_{+-} + B \cdot a_{--} = 0,$$

which, for generic c where its determinant is nonzero, has a unique solution

$$A = \frac{\sigma_2^p \cdot a_{--}}{\Delta} = \frac{1}{f(c)} \cdot a_{--}$$
$$B = \frac{-\sigma_2^p \cdot a_{+-}}{\Delta} = -\frac{1}{f(c)} \cdot a_{+-}.$$

This solution, which is a priori a rational function in $\mathfrak{h}^*$, is actually a polynomial $A, B \in S(\mathfrak{h}^*)$. So, whenever $f(c) \neq 0$, the vector $\sigma_2^p \otimes b_+$ is contained in $\langle v_+, v_- \rangle$.

The vector $\sigma_2^p \otimes b_-$ can be obtained from $\sigma_2^p \otimes b_+$ by the S_3 action, so it is also contained in $\langle v_+, v_- \rangle$.

3. Similarly, checking if the vector $\sigma_3^p \otimes b_+$ is in $\langle v_+, v_- \rangle$ ends up being equivalent to checking if there are $A, B \in S(\mathfrak{h}^*)$ such that

$$A \otimes b_+ + B \otimes b_- = \sigma_3^p \otimes b_+,$$

which has a unique (rational) solution

$$A = \frac{\sigma_3^p \cdot a_{--}}{f(c)\sigma_2^p}$$
$$B = \frac{-\sigma_3^p \cdot a_{+-}}{f(c)\sigma_2^p}.$$

These A, B are rational functions in $\mathfrak{h}^*$ and not elements of $S(\mathfrak{h}^*)$, and they are the only solutions when $f(c) \neq 0$. So, we conclude that $\sigma_3^p \otimes b_+$ and consequently $\sigma_3^p \otimes b_-$ do not lie in $\langle v_+, v_- \rangle$.

4. The Hilbert series of module $M_{1,c}(\texttt{stand})/\langle v_+, v_-, \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle$ can be calculated as

$$\mathrm{Hilb}_{M_{1,c}(\texttt{stand})}(z) - \mathrm{Hilb}_{\langle v_+,v_- \rangle}(z) - \mathrm{Hilb}_{\langle \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle}(z) + \mathrm{Hilb}_{\langle v_+,v_- \rangle \cap \langle \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle}(z).$$

We know that
$$\mathrm{Hilb}_{M_{1,c}(\texttt{stand})}(z) = \frac{2}{(1-z)^2}.$$

By part 1 of this Lemma, $\langle v_+, v_- \rangle \cong M_{1,c}(\texttt{stand})[-p]$, and similarly $\langle \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle \cong M_{1,c}(\texttt{stand})[-3p]$, so

$$\mathrm{Hilb}_{\langle v_+,v_- \rangle}(z) = \frac{2z^p}{(1-z)^2}, \quad \mathrm{Hilb}_{\langle \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle}(z) = \frac{2z^{3p}}{(1-z)^2}.$$

The remaining task is to describe $\langle v_+, v_- \rangle \cap \langle \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle$.

Let v be an arbitrary element of $\langle v_+, v_- \rangle \cap \langle \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle$. This means that there exist $A, B, C, D \in S(\mathfrak{h}^*)$ such that

$$v = Av_+ + Bv_- = C\sigma_3^p \otimes b_+ + D\sigma_3^p \otimes b_-. \tag{11.2.20}$$

As before we get

$$A = \frac{C \cdot a_{--} + D \cdot a_{-+}}{f(c)\sigma_2^p}\sigma_3^p$$
$$B = \frac{-C \cdot a_{+-} + D \cdot a_{++}}{f(c)\sigma_2^p}\sigma_3^p.$$

Given that σ_2 and σ_3 are algebraically independent so their powers are coprime, it follows that σ_2^p needs to divide $C \cdot a_{--} + D \cdot a_{-+}$ and $-C \cdot a_{+-} + D \cdot a_{++}$, and σ_3^p needs to divide both A and B. In other words, there are $A', B' \in S(\mathfrak{h}^*)$ such that

$$A = A' \cdot \sigma_3^p, \quad B = B' \cdot \sigma_3^p.$$

Rewriting equation (11.2.20) we get

$$v = A'\sigma_3^p v_+ + B'\sigma_3^p v_- = C\sigma_3^p \otimes b_+ + D\sigma_3^p \otimes b_-.$$

For any choice of $A', B' \in S(\mathfrak{h}^*)$ there are unique $C, D \in S(\mathfrak{h}^*)$ satisfying this, given by

$$C = A'a_{++} + B'a_{-+}$$
$$D = A'a_{+-} + B'a_{--}.$$

This lets us conclude that v is in the submodule generated by $\sigma_3^p v_+$ and $\sigma_3^p v_-$, and that

$$\langle v_+, v_- \rangle \cap \langle \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle = \langle \sigma_3^p v_+, \sigma_3^p v_- \rangle.$$

By a similar argument as before, this module is isomorphic to $M_{1,c}(\texttt{stand})[-4p]$, its Hilbert series is

$$\mathrm{Hilb}_{\langle \sigma_3^p v_+, \sigma_3^p v_- \rangle}(z) = \frac{2z^{4p}}{(1-z)^2},$$

and thus the Hilbert series of $M_{1,c}(\texttt{stand})/\langle v_+, v_-, \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle$ equals

$$\frac{2}{(1-z)^2} - \frac{2z^p}{(1-z)^2} - \frac{2z^{3p}}{(1-z)^2} + \frac{2z^{4p}}{(1-z)^2} = 2\frac{(1-z^p)(1-z^{3p})}{(1-z)^2}.$$

5. By the results of [DeSa14] discussed in Section 5.3, the Hilbert polynomial of the module $L_{1,c}(\texttt{stand})$ for generic c equals $2\frac{(1-z^p)(1-z^{3p})}{(1-z)^2}$. The module $L_{1,c}(\texttt{stand})$ is a quotient of $M_{1,c}(\texttt{stand})/\langle v_+, v_-, \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle$, which itself has the same Hilbert polynomial. So, we conclude that the singular vectors we found are indeed all the singular vectors,

and

$$M_{1,c}(\texttt{stand})/\langle v_+, v_-, \sigma_3^p \otimes b_+, \sigma_3^p \otimes b_- \rangle = L_{1,c}(\texttt{stand})$$

is irreducible.

□

Chapter 12

Irreducible Representations of $H_{t,c}(S_3, \mathfrak{h})$ in Characteristic $p > 3$ for Special c

We now turn our attention to special values of the parameter c, which are not covered by Theorem 11.0.1. The value of the characteristic remains $p > 3$. In this case, when $t = 0$ the only special value of c is $c = 0$, and when $t = 1$, the special values of c are $c \in \{0, 1, 2 \ldots, p-1\}$. We will sometimes consider these c as elements of $\mathbb{Z}$ to allow us to write inequalities such as $0 < c < p/6$. At other times (when they are coefficients in the algebra relations), we consider them as elements of $\Bbbk$ as before.

The aim of this chapter is to prove the following theorem.

Theorem 12.0.1. *The irreducible representations $L_{t,c}(\tau)$ of the rational Cherednik algebra $H_{t,c}(S_3)$ over an algebraically closed field of characteristic $p > 3$, for special c, $t = 0, 1$, and any τ, are described by the following tables.*

Characters:

$\boldsymbol{p > 3}$	$\tau = \texttt{triv}$
$t = 0,\ c = 0$	$[\texttt{triv}]$
$t = 1,\ c = 0$	$\chi_{S^{(p)}(\mathfrak{h}^*)}(z)$
$t = 1$, $0 < c < p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{triv}] - [\texttt{stand}]z^{3c+p} + [\texttt{sign}]z^{2(3c+p)}\big)$
$t = 1$, $p/3 < c < p/2$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{triv}] - [\texttt{stand}]z^{3c-p} + [\texttt{sign}]z^{2(3c-p)}\big)$
$t = 1$, $p/2 < c < 2p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{triv}] - [\texttt{sign}]z^{6c-3p}\big)(1 - z^p)$
$t = 1$, $2p/3 < c < p$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{triv}] - [\texttt{stand}]z^{3c-2p} + [\texttt{sign}]z^{2(3c-2p)}\big)$

$\boldsymbol{p > 3}$	$\tau = \texttt{sign}$
$t = 0,\ c = 0$	$[\texttt{sign}]$
$t = 1,\ c = 0$	$\chi_{S^{(p)}(\mathfrak{h}^*)}(z) \cdot [\texttt{sign}]$
$t = 1$, $0 < c < p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{sign}] - [\texttt{stand}]z^{p-3c} + [\texttt{triv}]z^{2(p-3c)}\big)$
$t = 1$, $p/3 < c < p/2$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{sign}] - [\texttt{triv}]z^{-6c+3p}\big)(1 - z^p)$
$t = 1$, $p/2 < c < 2p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot ([\texttt{sign}] - [\texttt{stand}]z^{2p-3c} + [\texttt{triv}]z^{2(2p-3c)})$
$t = 1$, $2p/3 < c < p$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{sign}] - [\texttt{stand}]z^{4p-3c} + [\texttt{triv}]z^{2(4p-3c)}\big)$

$\boldsymbol{p > 3}$	$\tau = \texttt{stand}$
$t = 0,\ c = 0$	$[\texttt{stand}]$
$t = 1,\ c = 0$	$\chi_{S^{(p)}(\mathfrak{h}^*)}(z) \cdot [\texttt{stand}]$
$t = 1$, $0 < c < p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot ([\texttt{stand}] - [\texttt{triv}]z^{p-3c} - [\texttt{stand}]z^{p} - [\texttt{sign}]z^{p+3c} + 2[\texttt{sign}]z^{2p})$
$t = 1$, $p/3 < c < p/2$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{stand}] - [\texttt{sign}]z^{-p+3c} - [\texttt{triv}]z^{3p-3c} - [\texttt{sign}]z^{p+3c} - [\texttt{triv}]z^{5p-3c} + [\texttt{stand}]z^{4p}\big)$
$t = 1$, $p/2 < c < 2p/3$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot \big([\texttt{stand}] - [\texttt{triv}]z^{-3c+2p} - [\texttt{sign}]z^{3c} - [\texttt{triv}]z^{-3c+4p} - [\texttt{sign}]z^{3c+2p} + [\texttt{stand}]z^{4p}\big)$
$t = 1$, $2p/3 < c < p$	$\chi_{S(\mathfrak{h}^*)}(z) \cdot ([\texttt{stand}] - [\texttt{sign}]z^{3c-2p} - [\texttt{stand}]z^{p} - [\texttt{triv}]z^{4p-3c} + 2[\texttt{triv}]z^{2p})$

with

$$\chi_{S(\mathfrak{h}^*)}(z) = \frac{1}{(1-z^2)(1-z^3)}\left([\mathtt{triv}] + [\mathtt{stand}](z+z^2) + [\mathtt{sign}]z^3\right),$$
$$\chi_{S^{(p)}(\mathfrak{h}^*)}(z) = \chi_{S(\mathfrak{h}^*)}(z)\cdot(1-[\mathtt{stand}]z^p + [\mathtt{sign}]z^{2p}).$$

Hilbert polynomials:

$\boldsymbol{p>3}$	$\tau = \mathtt{triv}$	$\tau = \mathtt{sign}$
$t=0,\ c=0$	1	1
$t=1$		
$c=0$	$\left(\frac{1-z^p}{1-z}\right)^2$	$\left(\frac{1-z^p}{1-z}\right)^2$
$0<c<p/3$	$\left(\frac{1-z^{3c+p}}{1-z}\right)^2$	$\left(\frac{1-z^{p-3c}}{1-z}\right)^2$
$p/3<c<p/2$	$\left(\frac{1-z^{3c-p}}{1-z}\right)^2$	$\frac{(1-z^{3p-6c})(1-z^p)}{(1-z)^2}$
$p/2<c<2p/3$	$\frac{(1-z^{6c-3p})(1-z^p)}{(1-z)^2}$	$\left(\frac{1-z^{2p-3c}}{1-z}\right)^2$
$2p/3<c<p$	$\left(\frac{1-z^{3c-2p}}{1-z}\right)^2$	$\left(\frac{1-z^{4p-3c}}{1-z}\right)^2$
	[Li14] (partial)	

$\boldsymbol{p>3}$	$\tau = \mathtt{stand}$
$t=0,\ c=0$	2
$t=1$	
$c=0$	$2\left(\frac{1-z^p}{1-z}\right)^2$
$0<c<p/3$	$\frac{2-z^{p-3c}-2z^p-z^{p+3c}+2z^{2p}}{(1-z)^2}$
$p/3<c<p/2$	$\frac{2-z^{-p+3c}-z^{3p-3c}-z^{p+3c}-z^{5p-3c}+2z^{4p}}{(1-z)^2}$
$p/2<c<2p/3$	$\frac{2-z^{-3c+2p}-z^{3c}-z^{-3c+4p}-z^{3c+2p}+2z^{4p}}{(1-z)^2}$
$2p/3<c<p$	$\frac{2-z^{3c-2p}-2z^p-z^{4p-3c}+2z^{2p}}{(1-z)^2}$

In all cases, the singular vectors are known explicitly and are calculated by us for c in the range $p/2 < c < 2p/3$ and otherwise given by [Li14]. The character formulas are not provided by [Li14] but they are easily calculated from the singular vectors so we include them for completeness. When $\tau =$ `stand` we rely on the minor technical assumption 12.2.1 for c in the range $p/6 < c < p/3$, and the results for c in the range $p/3 < c < 2p/3$ are conjectural.

Proof. For all τ and t, the case $c = 0$ is standard; when $t = 0$, the result is explained in Proposition 2.6.11, while for $t = 1$ the result follows from Proposition 2.6.13 and Corollary 7.2.6.

For $\tau =$ `triv`, the remaining cases of $t = 1$, $c \neq 0 \in \mathbb{F}_p$ fall into several cases depending on where c lies in the set $\{0, 1, \ldots, p-1\}$. The paper [Li14] deals with all these intervals except one, $p/2 < c < 2p/3$, where they give conjectured degrees of the generators. The work of [Li14] can be found in Section 5.2 and we deal with the remaining case in Section 12.1.

For $\tau =$ `sign`, the character formulas follow from the character formulas for `triv` by Corollary 2.8.3.

For $\tau =$ `stand` $t = 1$ and $c \in \mathbb{F}_p$, the case $0 < c < p/3$ is done with all the proofs in Section 12.2, the case $p/3 < c < p/2$ is stated conjecturally and with no proofs at the end of that section. The cases $p/2 < c < p$ follow from them using Corollary 2.8.3 and the fact that `stand` $\otimes$ `sign` $\cong$ `stand`. □

12.1 The irreducible representation $L_{1,c}(\mathtt{triv})$ characteristic $p > 3$ for special c

The paper [Li14] deals with some special values for c when $\tau =$ `triv`. Here we restate Theorem 3.3 of that paper in our conventions.

Proposition 12.1.1 ([Li14], Theorem 3.3)**.** *Consider the rational Cherednik algebra $H_{1,c}(S_3, \mathfrak{h})$ over an algebraically closed field of characteristic $p > 3$, for $t = 1$ and $c \in \mathbb{F}_p$. The degrees of the generators of the maximal proper graded submodule of the Verma module $M_{1,c}(\mathtt{triv})$ are as follows:*

1. $0 < c < p/3$*: two generators of degree* $3c + p$,
2. $p/3 < c < p/2$*: two generators of degree* $3c - p$,
3. $2p/3 < c < p$*: two generators of degree* $3c - 2p$.

In all these cases the generators of this maximal graded submodule are known explicitly (see the proof of [Li14] Theorem 3.3), the quotient of $M_{1,c}(\mathtt{triv}) \cong S(\mathfrak{h}^*)$ by the maximal proper *graded submodule is a complete intersection, and the character and Hilbert polynomial of the irreducible quotient* $L_{1,c}(\texttt{triv})$ *is*

$$\chi_{L_{1,c}(\texttt{triv})}(z) = \chi_{S(\mathfrak{h}^*)}(z) \cdot (1 - z^d[\texttt{stand}] + z^{2d}[\texttt{sign}]),$$
$$Hilb_{L_{1,c}(\texttt{triv})}(z) = \left(\frac{1-z^d}{1-z}\right)^2,$$

for d *the degree of the generators as listed above.*

Remark 12.1.2. Note some differences from the statement here and the way this theorem is stated in [Li14]. The first one is that [Li14] works with the reflection representation V and we work with $\mathfrak{h}$; the dictionary for translating results from one setting to the other is given in Proposition 4.2.6 and we therefore list one generator fewer (the generator is σ_1^p, which is 0 in our setting). Next, there is a typo in the statement of Case (3) of [Li14]; the correct degrees are listed here and the proof in [Li14] is correct. We also calculate the characters, which is straightforward from the information provided in [Li14]. The results of [Li14] are missing in one case, namely the $p/2 < c < 2p/3$, where they conjecturally give the degrees of the generators, and which we address in detail in this section, calculating the generators of the maximal proper graded submodule, calculating the character of the quotient, and showing the quotient is irreducible (thus proving their conjecture). Finally, we want to point out that the results of [Li14] in the cases $0 < c < p/2$ and $2p/3 < c < p$ lack three steps in the proofs: a minor step showing, in their notation, that $D_2(G_1) = 0$ (which is completely straightforward), the proof that the quotient is a complete intersection and thus its Hilbert polynomial is as stated (which is explicitly assumed in their Theorem 3.3), and the proof that the quotient is irreducible (which relies on the yet unpublished results by Roman Bezrukavnikov and Andrei Okounkov). We do not do this here, but note that these results all appear correct and can be verified using similar methods to our work in the case of $p/2 < c < 2p/3$.

The aim of this section is to describe $L_{1,c}(\texttt{triv})$ in the non-modular case $p > 3$ for those values of special c that [Li14] does not cover, namely $p/2 < c < 2p/3$. We fix those values of the parameter throughout this section. The main result is the following theorem.

Theorem 12.1.3. *The irreducible representation* $L_{1,c}(\texttt{triv})$ *of the rational Cherednik algebra* $H_{1,c}(S_3, \mathfrak{h})$ *over an algebraically closed field of characteristic* $p > 3$ *for* $c \in \mathbb{F}_p$, $p/2 < c < 2p/3$, *is the quotient of the Verma module* $M_{1,c}(\texttt{triv})$ *by the submodule generated by*

$$v_{6c-3p} = q^{2c-p} = (-4\sigma_2^3 - 27\sigma_3^2)^{c-\frac{p+1}{2}} q$$
$$v_p = \sigma_2^{2p-3c}\sigma_3 \sum_{j=0}^{c-\frac{p+1}{2}} \binom{c-\frac{p+1}{2}}{j} \frac{1}{2j+1} (-4\sigma_2^3)^{c-\frac{p+1}{2}-j} (-27\sigma_3^2)^j.$$

This quotient is a complete intersection, and its character and Hilbert polynomial are

$$\chi_{L_{1,c}(\texttt{triv})}(z) = \chi_{S(\mathfrak{h}^*)}(z)(1 - [\texttt{sign}]z^{6c-3p} - [\texttt{triv}]z^p + [\texttt{sign}]z^{6c-2p})$$
$$Hilb_{L_{1,c}(\texttt{triv})}(z) = \frac{(1 - z^{6c-3p})(1 - z^p)}{(1 - z)^2}.$$

Proof. First recall that Lemma 10.0.2 tells us the Casimir element $\Omega \in H_{1,c}(S_3, \mathfrak{h})$ acts on an irreducible S_3 subrepresentation τ of some Verma module (or its quotient) which consists of singular vectors by a scalar depending on τ as follows:

$$\Omega|_{\texttt{triv}} = 0 \cdot \mathrm{id}, \quad \Omega|_{\texttt{sign}} = 6c \cdot \mathrm{id}, \quad \Omega|_{\texttt{stand}} = 3c \cdot \mathrm{id}. \tag{12.1.4}$$

As a consequence, singular vectors in $M_{1,c}(\texttt{triv})$ and its quotients only appear in degrees of the form kp, $6c + kp$ and $3c + kp$ for some $k \in \mathbb{Z}$. Furthermore, singular vectors in degrees of the form kp form S_3 subrepresentations of type `triv`, singular vectors in degrees of the form $6c + kp$ form S_3 subrepresentations of type `sign`, and singular vectors in degrees of the form $3c + kp$ form S_3 subrepresentations of type `stand`.

Second, we note that for $c \in \mathbb{F}_p$ with $\frac{p}{2} < c < \frac{2p}{3}$ we have

$$0 < 6c - 3p < 3c - p < p, \tag{12.1.5}$$

and these are the only integers of the form $kp, 3c + kp$ and $6c + kp$ with $k \in \mathbb{Z}$ in the interval $[0, p]$.

These two facts together now spell out a strategy for describing $L_{1,c}(\texttt{triv})$: starting from the lowest degrees and going higher, we test for singular vectors. The first fact tells us in which degrees can the singular vectors appear, and of what isotypic component are they. This allows us to use the basis from Theorem 7.2.2 to reduce the size of the space where we are looking for the singular vectors, by only looking at specific isotypic components in specific degrees. The second fact tells us in which order to look at these graded pieces. This order is important because as soon as we take a quotient by some submodule in some degree, all calculations in higher degrees need to be done in the quotient module and no longer in the Verma module. This is because there could be singular vectors in the quotient of the Verma module which do not lift to a singular vector in the Verma module (see Lemma 12.1.20).

By equations (12.1.4) and (12.1.5), the first space to consider is a `sign` isotypic component in degree $6c - 3p$. In Lemma 12.1.6 we show that

$$v_{6c-3p} = q^{2c-p} = (-4\sigma_2^3 - 27\sigma_3^2)^{c - \frac{p+1}{2}} q$$

is a singular vector in degree $6c - 3p$, generating a submodule of $M_{1,c}(\texttt{triv})$ isomorphic to $M_{1,c}(\texttt{sign})$. In Lemma 12.1.7 we show there are no other singular vectors in degree $6c - 3p$.

Next, we take the quotient $M_{1,c}(\texttt{triv})/\left\langle q^{2c-p} \right\rangle$ and continue examining the degrees as dictated by equations (12.1.4) and (12.1.5). In Lemma 12.1.8 we show that there are no singular vectors in degree $3c-p$ of the module $M_{1,c}(\texttt{triv})/\left\langle q^{2c-p}\right\rangle$. In Lemma 12.1.20 we show that there is a 1-dimensional space of singular vectors in degree p of the module $M_{1,c}(\texttt{triv})/\langle q^{2c-p}\rangle$, in the $\texttt{triv}$ isotypic component and spanned by

$$v_p=\sigma_2^{2p-3c}\sigma_3\sum_{j=0}^{c-\frac{p+1}{2}}\binom{c-\frac{p+1}{2}}{j}\frac{1}{2j+1}(-4\sigma_2^3)^{c-\frac{p+1}{2}-j}(-27\sigma_3^2)^j.$$

We note that v_p is only singular in the quotient $M_{1,c}(\texttt{triv})/\left\langle q^{2c-p}\right\rangle$, and is not an image of a singular vector in $M_{1,c}(\texttt{triv})$.

In Lemma 12.1.23 we show that the quotient of $M_{1,c}(\texttt{triv})$ by the submodule generated by v_{6c-3p} and v_p is a complete intersection and calculate its character and Hilbert polynomial.

Its Hilbert polynomial turns out to be

$$\mathrm{Hilb}_{L_{1,c}(\texttt{triv})}(z)=\frac{(1-z^{6c-3p})(1-z^p)}{(1-z)^2}.$$

By equations (12.1.4) and (12.1.5), the next degree to consider and search for singular vectors is $6c-2p$. However, the above Hilbert polynomial shows that the maximal degree of $M_{1,c}(\texttt{triv})/\left\langle v_{6c-3p},v_p\right\rangle$ is $6c-3p-1+p-1=6c-2p-2<6c-2p$. This means the quotient $M_{1,c}(\texttt{triv})/\left\langle v_{6c-3p},v_p\right\rangle$ has no singular vectors, and is thus irreducible and equal to $L_{1,c}(\texttt{triv})$. □

We now proceed with the details of the above proof, listed as a sequence of lemmas. We keep the assumptions on the parameters t,c,p that were listed in Theorem 12.1.3.

Lemma 12.1.6. *The vector*

$$v_{6c-3p}=q^{2c-p}=(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}}q$$

is singular in $M_{1,c}^{6c-3p}(\texttt{triv})$, *and generates a subrepresentation* $\langle v_{6c-3p}\rangle$ *isomorphic to* $M_{1,c}(\texttt{sign})[-6c+3p]$.

Proof. The vector v_{6c-3p} satisfies $(12).v_{6c-3p}=-v_{6c-3p}$ so by Lemma 10.0.4, it is enough to calculate $D_{y_1}(v_{6c-3p})$ to see if it is singular. We check this directly, using computations from Chapter 10 and the fact that q^{2c-p-1} is a symmetric polynomial.

$$\begin{aligned}D_{y_1}(q^{2c-p})&=(2c-p)q^{2c-p-1}\partial_{y_1}(q)-cq^{2c-p-1}\frac{\mathrm{id}-(12)}{x_1-x_2}(q)-cq^{2c-p-1}\frac{\mathrm{id}-(13)}{x_1-x_3}(q)\\&=cq^{2c-p-1}\Bigg(2\cdot\frac{-1}{4}(-b_+^2+3b_-^2)+2\cdot\frac{1}{4}(2b_+b_-)+2\sigma_2\\&\qquad+\frac{1}{3}(-b_+^2+3b_-^2)-2\sigma_2+\frac{1}{6}(-b_+^2+3b_-^2)-\frac{1}{2}\cdot 2b_+b_-\Bigg)=0.\end{aligned}$$

This shows v_{6c-3p} is singular, so it induces a map

$$\varphi : M_{1,c}(\texttt{sign})[-6c+3p] \to M_{1,c}(\texttt{triv}).$$

Seen as a map $S(\mathfrak{h}^*) \cong M_{1,c}(\texttt{sign}) \to M_{1,c}(\texttt{triv}) \cong S(\mathfrak{h}^*)$ this map is just multiplication by q^{2c-p} which is injective. So, v_{6c-3p} generates a subrepresentation $\langle v_{6c-3p}\rangle$ of $M_{1,c}(\texttt{triv})$ isomorphic to $M_{1,c}(\texttt{sign})[-6c+3p]$.

□

Lemma 12.1.7. *The only singular vectors in $M_{1,c}^{6c-3p}(\texttt{triv})$ are multiples of v_{6c-3p}.*

Proof. By Theorem 7.2.2, a basis for the sign isotypic component in degree $6c - 3p$ is

$$\{\sigma_2^a\sigma_3^b \cdot q \mid 2a + 3b + 3 = 6c - 3p\}.$$

By the results in Chapter 10, we have that

$$D_{y_1}(\sigma_2^a\sigma_3^b \cdot q) = \left(\frac{-3}{2}a\sigma_2^{a-1}\sigma_3^{b+1} + \frac{1}{3}b\sigma_2^{a+2}\sigma_3^{b-1}\right)(b_+ - b_-) + \frac{-(2a+3b+3-6c)}{12}\sigma_2^a\sigma_3^b((-b_+^2 + 3b_-^2) - 2b_+b_-).$$

Notice that in our case we have $2a + 3b + 3 = 6c - 3p$, so this reduces to

$$D_{y_1}(\sigma_2^a\sigma_3^b \cdot q) = \left(\frac{-3}{2}a\sigma_2^{a-1}\sigma_3^{b+1} + \frac{1}{3}b\sigma_2^{a+2}\sigma_3^{b-1}\right)(b_+ - b_-).$$

Now let us parametrise all a, b such that $2a + 3b + 3 = 6c - 3p$. Notice that $a = 3i$ for some $0 \le i \le c - \frac{p+1}{2}$, in which case $b = 2c - p - 1 - 2i$. This tells us we are looking for all singular vectors of the form

$$\sum_{i=0}^{c-\frac{p+1}{2}} \alpha_i\sigma_2^{3i}\sigma_3^{2c-p-1-2i} \cdot q,$$

and they will be singular if and only if

$$\sum_{i=0}^{c-\frac{p+1}{2}} \alpha_i \left(\frac{-9i}{2}\sigma_2^{3i-1}\sigma_3^{2c-p-2i} + \frac{1}{3}(2c-p-1-2i)\sigma_2^{3i+2}\sigma_3^{2c-p-2i-2}\right) = 0.$$

Noting that the first summand is zero for $i = 0$, we can rewrite this sum as

$$\sum_{i=0}^{c-\frac{p+1}{2}} \left(\alpha_{i+1}\frac{-9(i+1)}{2}\sigma_2^{3i+2}\sigma_3^{2c-p-2i-2} + \alpha_i\frac{1}{3}(2c-p-1-2i)\sigma_2^{3i+1}\sigma_3^{2c-p-2i-2}\right) = 0$$

and which is equivalent to

$$\alpha_{i+1}\frac{-9(i+1)}{2} + \alpha_i\frac{2}{3}\left(c - \frac{p+1}{2} - i\right) = 0 \quad 0 \le i \le c - \frac{p+1}{2}.$$

This can be further rewritten as

$$\alpha_{i+1} = \frac{2^2}{3^3}\frac{(c - \frac{p+1}{2} - i)}{i+1}\alpha_i \quad 0 \le i \le c - \frac{p+1}{2},$$

which is a system whose unique (up to overall scaling) solution is

$$\alpha_i = (-1)^{c-\frac{p+1}{2}} 4^i \cdot 27^{c-\frac{p+1}{2}-i} \cdot \binom{c - \frac{p+1}{2}}{i}, \quad 0 \le i \le c - \frac{p+1}{2}.$$

This shows that the unique (up to scaling) singular vector in degree $6c - 3p$ is

$$\begin{aligned} v_{6c-3p} &= (-1)^{c-\frac{p+1}{2}} \sum_{i=0}^{c-\frac{p+1}{2}} \binom{c - \frac{p+1}{2}}{i} 4^i \sigma_2^{3i} \cdot 27^{c-\frac{p+1}{2}-i} \sigma_3^{2(c-\frac{p+1}{2}-i)} \cdot q \\ &= (-4\sigma_2^3 - 27\sigma_3^2)^{c-\frac{p+1}{2}} \cdot q \\ &= q^{2c-p}. \end{aligned}$$

□

Lemma 12.1.8. *There are no singular vectors in degree $3c - p$ of $M_{1,c}(\texttt{triv})/\langle v_{6c-3p}\rangle$.*

Proof. It is straightforward to show that there are no singular vectors in degree $3c-p$ of the module $M_{1,c}(\texttt{triv})$, using a similar computation to the above and the basis

$$\{\sigma_2^a\sigma_3^b b_+ \mid 2a + 3b + 1 = 3c - p\} \cup \{\sigma_2^a\sigma_3^b(-b_+^2 + 3b_-^2) \mid 2a + 3b + 2 = 3c - p\}, \qquad (12.1.9)$$

of the S_2 invariant part of the `stand` isotypic component of $M_{1,c}^{3c-p}(\texttt{triv})$. However, this lemma claims more - that there are no singular vectors modulo $v_{6c-3p} = q^{2c-p}$. The basis (12.1.9) is not well suited to taking this quotient and the calculations are more involved, so we first change basis to the one in which taking this quotient becomes very easy. We then prove the lemma by direct computation. We distinguish two cases, depending on the remainder of p modulo 3.

Case 1. $p \equiv 2 \pmod 3$ The new basis of the S_2 invariant part of the `stand` isotypic component of $M_{1,c}^{3c-p}(\texttt{triv})$ we propose to use in this case is

$$\left\{q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i} \cdot b_+ \mid 0 \le i \le \frac{c}{2} - \frac{p+1}{6}\right\} \bigcup \left\{q^{2i+1}\sigma_3^{c-\frac{p+1}{3}-2i-1} \cdot b_- \mid 0 \le i \le \frac{c}{2} - \frac{p+1}{6} - \frac{1}{2}\right\}. \qquad (12.1.10)$$

This set is indeed a basis as it lies in the S_2 invariant part of the `stand` isotypic component of $M_{1,c}^{3c-p}(\mathtt{triv})$, is linearly independent and has the correct number of elements

After taking the quotient by $\langle q^{2c-p}\rangle$ the set (12.1.10) is reduced to

$$\left\{q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i}\cdot b_+ \mid 0\le i\le c-\frac{p+1}{2}\right\}\bigcup\left\{q^{2i+1}\sigma_3^{c-\frac{p+1}{3}-2i-1}\cdot b_- \mid 0\le i\le c-\frac{p+1}{2}-1\right\}. \tag{12.1.11}$$

This set spans the S_2 invariant part of the $\mathtt{stand}$ isotypic component of $M_{1,c}^{3c-p}(\mathtt{triv})/\langle q^{2c-p}\rangle$ and contains $2c-p$ elements. On the other hand, the multiplicity of $\mathtt{stand}$ in $M_{1,c}^{3c-p}(\mathtt{triv})$ is $\frac{3c-p+2}{3}$, the multiplicity of $\mathtt{stand}$ in $M_{1,c}^{3c-p}(\mathtt{sign})[-6c+3p]$ is $\frac{2p-3c+2}{3}$, so the multiplicity of $\mathtt{stand}$ in $M_{1,c}^{3c-p}(\mathtt{triv})/\langle q^{2c-p}\rangle$ is

$$\frac{3c-p+2}{3}-\frac{2p-3c+2}{3}=2c-p.$$

This shows that the set (12.1.11) is a basis of the S_2 invariant part of the $\mathtt{stand}$ isotypic component of $M_{1,c}^{3c-p}(\mathtt{triv})/\langle q^{2c-p}\rangle$.

We calculate the Dunkl operators in this basis.

$$\begin{aligned}D_{y_1}(q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i}\cdot b_+)=&\left(1-\frac{3c+p+1}{2}\right)q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i}\\&+\frac{1}{2}\left(c-\frac{p+1}{3}-2i\right)q^{2i+1}\sigma_3^{c-\frac{p+1}{3}-2i-1}+(-27i)q^{2i-1}\sigma_3^{c-\frac{p+1}{3}-2i+1}\\&+(-3i)q^{2i-1}\sigma_3^{c-\frac{p+1}{3}-2i}\sigma_2b_++\frac{1}{6}\left(c-\frac{p+1}{3}-2i\right)q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i-1}\sigma_2b_+\\&+(-3i)q^{2i-1}\sigma_3^{c-\frac{p+1}{3}-2i}\sigma_2b_--\frac{1}{2}\left(c-\frac{p+1}{3}-2i\right)q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i-1}\sigma_2b_-\end{aligned} \tag{12.1.12}$$

$$\begin{aligned}&D_{y_1}(q^{2i+1}\sigma_3^{c-\frac{p+1}{3}-2i-1}\cdot b_-)=\\&=\frac{-1}{6}\left(c-\frac{p+1}{3}-2i-1\right)q^{2i+2}\sigma_3^{c-\frac{p+1}{3}-2i-2}+\frac{9}{2}(2i+1-2c)q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i}\\&+\left(1+\frac{3}{2}\left(c-\frac{p+1}{3}\right)\right)q^{2i+1}\sigma_3^{c-\frac{p+1}{3}-2i-1}\\&+\left(\frac{-1}{2}(2i+1)+c\right)q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i-1}\sigma_2b_+-\frac{1}{6}\left(c-\frac{p+1}{3}-2i-1\right)q^{2i+1}\sigma_3^{c-\frac{p+1}{3}-2i-2}\sigma_2b_+\\&+\left(\frac{3}{2}(2i+1)-3c\right)q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i-1}\sigma_2b_--\frac{1}{6}\left(c-\frac{p+1}{3}-2i-1\right)q^{2i+1}\sigma_3^{c-\frac{p+1}{3}-2i-2}\sigma_2b_-\end{aligned} \tag{12.1.13}$$

Assume that

$$w=\sum_{i=0}^{c-\frac{p+1}{2}}\alpha_i q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i}\cdot b_+ +\sum_{i=0}^{c-\frac{p+1}{2}-1}\beta_i q^{2i+1}\sigma_3^{c-\frac{p+1}{3}-2i-1}\cdot b_-$$

is a singular vector. Using (12.1.12) and (12.1.13), we get that the `triv` component of $D_{y_1}(w)$ is 0 if and only if

$$\begin{aligned}&\sum_{i=0}^{c-\frac{p+1}{2}}\alpha_i\left(1-\frac{3c+p+1}{2}\right)q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i}\\&+\sum_{i=0}^{c-\frac{p+1}{2}-1}\beta_i\frac{-1}{6}\left(c-\frac{p+1}{3}-2i-1\right)q^{2i+2}\sigma_3^{c-\frac{p+1}{3}-2i-2}\\&+\sum_{i=0}^{c-\frac{p+1}{2}-1}\beta_i\frac{9}{2}(2i+1-2c)q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i}=0.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}&\sum_{i=0}^{c-\frac{p+1}{2}}\alpha_i\left(1-\frac{3c+p+1}{2}\right)q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i}\\&+\sum_{i=1}^{c-\frac{p+1}{2}}\beta_{i-1}\frac{-1}{6}\left(c-\frac{p+1}{3}-2i+1\right)q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i}\\&+\sum_{i=0}^{c-\frac{p+1}{2}-1}\beta_i\frac{9}{2}(2i+1-2c)q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i}=0.\end{aligned}$$

When $i=0$ this leads to

$$\alpha_0\cdot\left(1-\frac{3c+p+1}{2}\right)+\beta_0\cdot\frac{9}{2}(1-2c)=0, \tag{12.1.14}$$

while for $i=1,\dots,c-\frac{p+1}{2}-1$ we get

$$\alpha_i\cdot\left(1-\frac{3c+p+1}{2}\right)+\beta_i\cdot\frac{9}{2}(2i+1-2c)+\beta_{i-1}\cdot\frac{1}{6}\left(2i-1-c+\frac{p+1}{3}\right)=0 \tag{12.1.15}$$

and finally when $i=c-\frac{p+1}{2}$,

$$\alpha_{c-\frac{p+1}{2}}\cdot\left(1-\frac{3c+p+1}{2}\right)+\beta_{c-\frac{p+1}{2}-1}\cdot\frac{1}{6}\left(c-\frac{2(p+1)}{3}-1\right)=0. \tag{12.1.16}$$

Similarly, using (12.1.12) and (12.1.13), we get that the `sign` component of $D_{y_1}(w)$ is 0 if and only if for $i = 0$,

$$\alpha_0 \cdot \left(c - \frac{p+1}{3}\right) + \beta_0 \cdot \left(1 + \frac{3}{2}(c - \frac{p+1}{2})\right) + \alpha_1 \cdot (-27) = 0, \tag{12.1.17}$$

and for $i = 1, \dots, c - \frac{p+1}{2} - 1$,

$$\alpha_i \cdot \left(c - \frac{p+1}{3} - 2i\right) + \beta_i \cdot \left(1 + \frac{3}{2}(c - \frac{p+1}{2})\right) + \alpha_{i+1} \cdot (-27(i+1)) = 0, \tag{12.1.18}$$

and for $i = c - \frac{p+1}{2}$,

$$\alpha_{c-\frac{p+1}{2}} \cdot \left(-c + \frac{2(p+1)}{3}\right) = 0. \tag{12.1.19}$$

The system of equations (12.1.14)-(12.1.19) can now be shown to have no nonzero solutions by using (12.1.19) to deduce $\alpha_{c-\frac{p+1}{2}} = 0$, (12.1.16) to deduce $\beta_{c-\frac{p+1}{2}-1} = 0$, then alternating (12.1.15) and (12.1.18) to show that $\beta_{i-1} = 0$ and $\alpha_i = 0$ for all $i = 1, \dots, c - \frac{p+1}{2} - 1$ and finally using either (12.1.14) or (12.1.17) to show that $\alpha_0 = 0$. The only constants we divide by in this calculation are $c - \frac{p+1}{3} - 2i$ and $c - \frac{p+1}{3} - 2i + 1$, which are never equal to 0 for these i and $\frac{p}{2} < c < \frac{2p}{3}$. This shows that the only solution to the equation $D_{y_1}(w) = 0$ in the S_2 invariant part of the stand isotypic component of $M_{1,c}^{3c-p}(\texttt{triv})/\langle q^{2c-p}\rangle$ is $w = 0$, proving the claim of the Lemma in Case 1.

Case 2. $p \equiv 1 \pmod 3$ A similar proof works. The basis of the S_2 invariant part of the stand isotypic component of $M_{1,c}^{3c-p}(\texttt{triv})/\langle q^{2c-p}\rangle$ is

$$\left\{q^{2i}\sigma_3^{c-\frac{p+2}{3}-2i}\cdot(-b_+^2+3b_-^2) \mid 0 \le i \le c-\frac{p+1}{2}\right\} \bigcup \left\{q^{2i+1}\sigma_3^{c-\frac{p+2}{3}-2i-1}\cdot 2b_+b_- \mid 0 \le i < c-\frac{p+1}{2}\right\}.$$

We calculate the Dunkl operators in this basis to be:

$$\begin{aligned} &D_{y_1}(q^{2i}\sigma_3^{c-\frac{p+1}{3}-2i} \cdot (-b_+^2 + 3b_-^2)) = \\ &= 2\left(c - \frac{p+2}{3} - 2i\right)q^{2i}\sigma_3^{c-\frac{p+2}{3}-2i-1}\sigma_2^2 + (-9i)q^{2i-1}\sigma_3^{c-\frac{p+1}{3}-2i}\sigma_2^2 + \\ &\quad + \text{ something in the stand component of } M_{1,c}^{3c-p-1}(\texttt{triv}) \end{aligned}$$

$$\begin{aligned} &D_{y_1}\left(q^{2i+1}\sigma_3^{c-\frac{p+1}{3}-2i-1} \cdot 2b_+b_-\right) = \\ &= 6(2i+1-2c)q^{2i}\sigma_3^{c-\frac{p+2}{3}-2i-1}\sigma_2^2 + 2(c - \frac{p+2}{3} - 2i - 1)q^{2i+1}\sigma_3^{c-\frac{p+1}{3}-2i-2}\sigma_2^2 \\ &\quad + \text{ something in the stand component of } M_{1,c}^{3c-p-1}(\texttt{triv}). \end{aligned}$$

Requiring that a vector

$$w=\sum_{i=0}^{c-\frac{p+1}{2}}\alpha_i q^{2i}\sigma_3^{c-\frac{p+2}{3}-2i}\cdot(-b_+^2+3b_-^2)+\sum_{i=0}^{c-\frac{p+1}{2}-1}\beta_i q^{2i+1}\sigma_3^{c-\frac{p+2}{3}-2i-1}\cdot 2b_+b_-$$

satisfies $D_{y_1}(w)=0$ leads to to a system of equations for α_i,β_i. By asking that the `triv` component of $D_{y_1}(w)$ is 0 we get

$$\alpha_{c-\frac{p+1}{2}}=0$$
$$\alpha_i\cdot 2\left(c-\frac{p+2}{3}-2i\right)+\beta_i\cdot 6(2i+1-2c)=0,\qquad i=0,\dots,c-\frac{p+1}{2}-1$$

and by asking that the `sign` component of $D_{y_1}(w)$ is 0 we get

$$\alpha_{i+1}\cdot(-9)(i+1)+\beta_i\cdot 2\left(c-\frac{p+2}{3}-2i-1\right)=0,\qquad i=0,\dots,c-\frac{p+1}{2}.$$

This system has a unique solution $\alpha_i=0,\beta_i=0$ for all i, so $w=0$. This proves the Lemma in Case 2. □

Lemma 12.1.20. *There is a 1-dimensional space of singular vectors in degree p of the module $M_{1,c}(\texttt{triv})/\langle q^{2c-p}\rangle$, in the `triv` isotypic component and spanned by*

$$v_p=\sigma_2^{2p-3c}\sigma_3\sum_{j=0}^{c-\frac{p+1}{2}}\binom{c-\frac{p+1}{2}}{j}\frac{1}{2j+1}(-4\sigma_2^3)^{c-\frac{p+1}{2}-j}(-27\sigma_3^2)^j.$$

Note that v_p is only singular modulo q^{2c-p}, and is not an image of a singular vector in $M_{1,c}(\texttt{triv})$.

Proof. We will be calculating modulo

$$q^{2c-p}=(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}}q.$$

The basis we are working with in `triv` isotypic component of $M_{1,c}^p(\texttt{triv})$ is

$$\{\sigma_2^a\sigma_3^b\mid 2a+3b=p,\ 0\le a,b\}.$$

The relations among those elements coming from taking the quotient by q^{2c-p} are

$$\sigma_2^a\sigma_3^b q^{2c-p+1}=\sigma_2^a\sigma_3^b(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}+1}=0,$$

for all a,b such that $2a+3b+6(c-\frac{p+1}{2}+1)=p$. In particular, in this quotient we can express any power of σ_3^b with $b\ge 2c-p+1$ in terms of polynomials of lower degree in σ_3. So, the basis of the triv isotypic component in degree p of the quotient $M_{1,c}(\texttt{triv})/\left\langle q^{2c-p}\right\rangle$ is

$$\{\sigma_2^a\sigma_3^b \mid 2a+3b=p,\ 0\le a,\ 0\le b<2c-p+1\}.$$

To parametrise the set of all such a,b, note that if $2a+3b=p$, then b has to be odd, so $b=2j+1$ and $a=\frac{p-3}{2}-3j$. The inequalities $0\le b<2c-p+1$ now become $0\le j$ and $j\le c-\frac{p+1}{2}$, and the inequality $0\le a$ becomes $0\le \frac{p-3}{2}-3j$ which is automatically satisfied for all $j=0,1,\dots,c-\frac{p+1}{2}$. So, an arbitrary vector in the triv isotypic component of $M_{1,c}^p(\texttt{triv})/\left\langle q^{2c-p}\right\rangle$ can be uniquely written as

$$v=\sum_{j=0}^{c-\frac{p+1}{2}}\alpha_j\sigma_2^{\frac{p-3}{2}-3j}\sigma_3^{2j+1}$$

for some $\alpha_j\in\Bbbk$ (depending on c).

Let us check when such a vector is singular. We have

$$D_{y_1}(\sigma_2^a\sigma_3^b)=\frac{-a}{6}\sigma_2^{a-1}\sigma_3^b(b_++3b_-)+\frac{b}{36}\sigma_2^a\sigma_3^{b-1}((-b_+^2+3b_-^2)+3\cdot 2b_+b_-).$$

This formula makes it clear that there will be no singular vectors in $M_{1,c}^p(\texttt{triv})$, but there still might be singular vectors in degree p of the quotient $M_{1,c}(\texttt{triv})/\langle q^{2c-p}\rangle$. We have:

$$\begin{aligned}
D_{y_1}(v)&=\sum_{j=0}^{c-\frac{p+1}{2}}\alpha_j\Bigg(\frac{-\left(\frac{p-3}{2}-3j\right)}{6}\sigma_2^{\frac{p-3}{2}-3j-1}\sigma_3^{2j+1}(b_++3b_-)\\
&\qquad\qquad+\frac{2j+1}{36}\sigma_2^{\frac{p-3}{2}-3j}\sigma_3^{2j}\left((-b_+^2+3b_-^2)+3\cdot 2b_+b_-\right)\Bigg)\\
&=\sum_{j=0}^{c-\frac{p+1}{2}}\alpha_j\frac{2j+1}{12}\sigma_2^{\frac{p-3}{2}-3j-1}\sigma_3^{2j}\left(3\sigma_3b_++\frac{1}{3}\sigma_2(-b_+^2+3b_-^2)\right)\\
&\qquad\qquad-\alpha_j\frac{2j+1}{12}\sigma_2^{\frac{p-3}{2}-3j-1}\sigma_3^{2j}\left(-9\sigma_3b_+-\sigma_2\cdot 2b_+b_-\right)\\
&=\sum_{j=0}^{c-\frac{p+1}{2}}\alpha_j\frac{2j+1}{12}\sigma_2^{\frac{p-3}{2}-3j-1}\sigma_3^{2j}qb_--\alpha_j\frac{2j+1}{12}\sigma_2^{\frac{p-3}{2}-3j-1}\sigma_3^{2j}qb_+\\
&=\frac{1}{12}\sigma_2^{2p-3c}\left(\sum_{j=0}^{c-\frac{p+1}{2}}\alpha_j(2j+1)\sigma_2^{3(c-\frac{p+1}{2}-j)}\sigma_3^{2j}\right)q\cdot(b_--b_+).
\end{aligned}$$

We now note that this is a multiple of

$$(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}}q=\sum_{j=0}^{c-\frac{p+1}{2}}\binom{c-\frac{p+1}{2}}{j}(-4\sigma_2^3)^{c-\frac{p+1}{2}-j}(-27\sigma_3^2)^j$$

if and only if (up to overall scaling) we have for all j

$$\alpha_j(2j+1)=\binom{c-\frac{p+1}{2}}{j}(-4)^{c-\frac{p+1}{2}-j}(-27)^j.$$

This proves there is (up to scalars) exactly one singular vector in the $\mathtt{triv}$ isotypic component of $M^p_{1,c}(\mathtt{triv})/\left\langle q^{2c-p}\right\rangle$ equal to

$$v_p=\sigma_2^{2p-3c}\sigma_3\sum_{j=0}^{c-\frac{p+1}{2}}\binom{c-\frac{p+1}{2}}{j}\frac{1}{2j+1}(-4\sigma_2^3)^{c-\frac{p+1}{2}-j}(-27\sigma_3^2)^j.$$

□

Remark 12.1.21. We can rewrite it as

$$v_p=\sigma_2^{2p-3c}\sigma_3\sum_{i=0}^{c-\frac{p+1}{2}}(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}-i}(-4\sigma_2^3)^i\frac{\prod_{j=0}^{i-1}(2m-2j)}{\prod_{j=0}^{i}(2m+1-2j)}$$

though it is not clear if we gain anything by rewriting it like this, or if we can find a closed formula for it.

Remark 12.1.22. When $c=\frac{1}{2}$, it lies in this interval because $c=\frac{p+1}{2}\in\mathbb{F}_p$ lies between $p/2$ and $2p/3$. For $c=\frac{p+1}{2}$ the above formula says

$$v_p=\sigma_2^{\frac{p-3}{2}}\sigma_3.$$

Compare this with the formula in [Li14] Remark 3.5.

Lemma 12.1.23. *The vectors v_{6c-3p} and v_p form a complete intersection. The character and the Hilbert polynomial of the quotient $M_{1,c}(\mathtt{triv})/\langle v_{6c-3p},v_p\rangle$ are*

$$\chi_{M_{1,c}(\mathtt{triv})/\langle v_{6c-3p},v_p\rangle}(z)=\chi_{S(\mathfrak{h}^*)}(z)(1-[\mathtt{sign}]z^{6c-3p}-[\mathtt{triv}]z^p+[\mathtt{sign}]z^{6c-2p})$$

$$Hilb_{M_{1,c}(\mathtt{triv})/\langle v_{6c-3p},v_p\rangle}(z)=\frac{(1-z^{6c-3p})(1-z^p)}{(1-z)^2}.$$

To prove Lemma 12.1.23 we will need two auxiliary facts, which will not be used elsewhere.

Lemma 12.1.24. *For any $m\in\mathbb{N}$ with $m<p$ we have*

$$\sum_{j=0}^{m}\binom{m}{j}\frac{1}{2j+1}(-1)^j\neq 0.$$

Proof. Set

$$f_m(z)=\sum_{j=0}^{m}\binom{m}{j}\frac{1}{2j+1}(-1)^jz^{2j+1}.$$

The claim is then that $f_m(1) \neq 0$.

Differentiating formally we get that

$$f'_m(z) = \sum_{j=0}^{m} \binom{m}{j} (-1)^j z^{2j} = (1-z^2)^m,$$

so

$$f_m(z) = \int (1-z^2)^m dz, \quad f_m(0) = 0$$

is the primitive function of $(1-z^2)^m$ with no constant term.

There is a recursive formula for evaluating this integral, given by

$$\int (1-z^2)^m dz = \frac{z(1-z^2)^m}{2m+1} + \frac{2m}{2m+1}\int (1-z^2)^{m-1} dz.$$

As $\frac{z(1-z^2)^m}{2m+1}|_{z=0} = 0$, we have

$$f_m(z) = \frac{z(1-z^2)^m}{2m+1} + \frac{2m}{2m+1} f_{m-1}(z),$$

and so

$$f_m(1) = 0 + \frac{2m}{2m+1} f_{m-1}(1)$$

and in particular (using $m<p$ and $p \neq 2$) we get $f_m(1) \neq 0$ if and only if $f_{m-1}(1) \neq 0$. Inductively, it remains to show that $f_1(1) \neq 0$, which is straightforward from

$$f_1(z) = \int 1 - z^2 dz = z - \frac{1}{3}z^3$$

and

$$f_1(1) = \frac{2}{3} \neq 0.$$

□

Before the next lemma, recall that $q^2 = -4\sigma_2^3 - 27\sigma_3^2$. and v_p is given in Lemma 12.1.20.

Lemma 12.1.25. *If $k \in \mathbb{N}$ and $A, B \in S(\mathfrak{h}^*)^{S_3}$ are symmetric polynomials such that*

$$A(-4\sigma_2^3 - 27\sigma_3^2)^k = Bv_p,$$

then there exists a symmetric polynomial C such that

$$A = Cv_p, \quad B = C(-4\sigma_2^3 - 27\sigma_3^2)^k.$$

Proof. This is a statement about symmetric polynomials, which, by the the fundamental theorem of symmetric polynomials form an algebra isomorphic to a polynomial algebra

$S(\mathfrak{h}^*)^{S_3} \cong \Bbbk[a, b]$ with the isomorphism $\sigma_2 \mapsto a, \sigma_3 \mapsto b$. Abusing notation slightly, the problem becomes about $A, B \in \Bbbk[a, b]$ such that

$$A(-4a^3 - 27b^2)^k = Ba^{2p-3c}b \sum_{j=0}^{c-\frac{p+1}{2}} \binom{c - \frac{p+1}{2}}{j} \frac{1}{2j+1} (-4a^3)^{c-\frac{p+1}{2}-j}(-27b^2)^j.$$

The polynomial $-4a^3 - 27b^2$ is irreducible. It does not divide the polynomial $v_p = a^{2p-3c}b \sum_{j=0}^{c-\frac{p+1}{2}} \binom{c-\frac{p+1}{2}}{j} \frac{1}{2j+1}(-4a^3)^{c-\frac{p+1}{2}-j}(-27b^2)^j$; to see that, plug in the values $a = -3$ and $b = 2$ to get that $-4(-3)^3 - 27 \cdot 2^2 = 0$, while by Lemma 12.1.24

$$v_p|_{a=-3,\, b=2} = (-3)^{2p-3c} \cdot 2 \cdot (4 \cdot 27)^{c-\frac{p+1}{2}} \sum_{j=0}^{c-\frac{p+1}{2}} \binom{c - \frac{p+1}{2}}{j} \frac{1}{2j+1} (-1)^j \neq 0.$$

So, $(-4a^3 - 27b^2)$ divides B, and proceeding inductively we get that $(-4a^3 - 27b^2)^k$ divides B. This means there exists $C \in \mathbb{K}[a, b]$ such that

$$B = (-4a^3 - 27b^2)^k C.$$

It now follows that

$$A = C \cdot a^{2p-3c}b \sum_{j=0}^{c-\frac{p+1}{2}} \binom{c - \frac{p+1}{2}}{j} \frac{1}{2j+1} (-4a^3)^{c-\frac{p+1}{2}-j}(-27b^2)^j.$$

Returning to the setup with σ_2, σ_3 instead of a, b we get the claim. □

Proof of Lemma 12.1.23. The claim that v_{6c-3p} and v_p form a complete intersection follows from the claim that they belong to a regular sequence in the regular local ring $S(\mathfrak{h}^*)$. A regular sequence is a sequence such that each element of the sequence is a non-zero-divisor in the quotient by the ideal generated by all previous elements of the sequence. We claim that in $S(\mathfrak{h}^*) \cong M_{1,c}(\texttt{triv})$ we have

$$\langle v_{6c-3p} \rangle \cap \langle v_p \rangle = \langle v_{6c-3p} v_p \rangle,$$

thus showing that they form a regular sequence.

Consider any vector $u \in \langle v_{6c-3p} \rangle \cap \langle v_p \rangle$. As we are in the non-modular case, by using S_3 projection formulas we can assume that u is in an isotypic component of type `triv`, `sign`, or `stand`. It is of the form

$$u = Av_{6c-3p} = Bv_p$$

for some $A,B\in S(\mathfrak{h}^*)$. Recall that

$$v_{6c-3p}=q^{2c-p}=(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}}q,$$

that v_{6c-3p} is in the `sign` isotypic component and that v_p is in the `triv` isotypic component.

Case 1. If the vector u is in the `sign` isotypic component then A is symmetric and B is antisymmetric. Write $B=B'\cdot q$ for some symmetric polynomial B'. The problem now becomes finding $A,B'\in S(\mathfrak{h}^*)^{S_3}$ such that

$$A(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}}=B'v_p$$

Using Lemma 12.1.25 we get that there exists a symmetric polynomial C such that

$$A=Cv_p,\ B'=C(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}},\ B=C(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}}q=Cq^{2c-p}$$

and so the vector $u=Cv_{6c-3p}v_p$ lies in $\langle v_{6c-3p}v_p\rangle$ as claimed.

Case 2. If the vector u is in the `triv` isotypic component then A is antisymmetric and B is symmetric. Write $A=A'\cdot q$ for some symmetric polynomial A', reducing the problem to

$$A'\cdot(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}+1}=Bv_p$$

with $A',B\in S(\mathfrak{h}^*)^{S_3}$. Using Lemma 12.1.25 again we get that there exists a symmetric polynomial C such that

$$A'=Cv_p,\ A=Cv_pq,\ B=C(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}+1}=Cq^{2c-p+1},$$

and that so the vector $u=Cv_{6c-3p}v_pq$ lies in $\langle v_{6c-3p}v_p\rangle$ as claimed.

Case 3. If the vector u is in the `stand` isotypic component, we can assume without loss of generality that it is S_2 invariant. Then A lies in the S_2 antiinvariant part of the `stand` isotypic component of $S(\mathfrak{h}^*)$, B in the S_2 invariant, and using the basis from Theorem 7.2.2 we can write

$$A=A_1b_-+A_2\cdot 2b_+b_-,\quad B=B_1b_++B_2(-b_+^2+3b_-^2)$$

for some $A_1,A_2,B_1,B_2\in S(\mathfrak{h}^*)^{S_3}$. The equation $Aq^{2c-p}=Bv_p$ becomes

$$(A_1b_-q+A_2\cdot 2b_+b_-q)(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}}=(B_1b_++B_2(-b_+^2+3b_-^2))v_p$$

or, using the results from Chapter 10,

$$\left(A_1\cdot 3\sigma_3b_++A_1\cdot\frac{1}{3}\sigma_2(-b_+^2+3b_-^2)+A_2\cdot 4\sigma_2^2b_+-A_2\cdot 3\sigma_3(-b_+^2+3b_-^2)\right)(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}}=$$

$$=\left(B_1b_++B_2(-b_+^2+3b_-^2)\right)v_p.$$

Rewriting further we get the system

$$\left(A_1\cdot 3\sigma_3 + A_2\cdot 4\sigma_2^2\right)(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}} = B_1 v_p$$
$$\left(A_1\cdot \frac{1}{3}\sigma_2 - A_2\cdot 3\sigma_3\right)(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}} = B_2 v_p,$$

which is again a problem about symmetric polynomials. We use Lemma 12.1.25 twice to get that there exist symmetric polynomials C_1, C_2 such that

$$A_1\cdot 3\sigma_3 + A_2\cdot 4\sigma_2^2 = C_1 v_p, \quad B_1 = C_1(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}}$$

$$A_1\cdot \frac{1}{3}\sigma_2 - A_2\cdot 3\sigma_3 = C_2 v_p, \quad B_2 = C_2(-4\sigma_2^3-27\sigma_3^2)^{c-\frac{p+1}{2}}.$$

From here we get

$$A_1\cdot(-4\sigma_2^3-27\sigma_3^2) = (-3)(C_1\cdot 3\sigma_3 + C_2\cdot 4\sigma_2^2)v_p$$
$$A_2\cdot(-4\sigma_2^3-27\sigma_3^2) = (-3)(C_1\cdot \frac{1}{3}\sigma_2^3 + C_2\cdot 3\sigma_3)v_p.$$

We now use Lemma 12.1.25 twice more to get that there exist symmetric polynomials D_1, D_2 such that

$$A_1 = D_1 v_p,\ A_2 = D_2 v_p.$$

Finally, it follows that the vector u $Aq^{2c-p} = Bu$ can be written as

$$u = Av_{pc-3p} = (D_1 b_- + D_2\cdot 2b_+b_-)v_{pc-3p}v_p,$$

which proves it in $\langle v_{pc-3p}v_p\rangle$ as claimed.

The computation of the character and the Hilbert series is now standard, using that $L_{1,c}(\texttt{triv})$ is a quotient of $M_{1,c}(\texttt{triv})$ by the sum of the submodule generated by v_{6c-3p} isomorphic to $M_{1,c}(\texttt{sign})[-6c+3p]$ and the submodule generated by v_p isomorphic to $M_{1,c}(\texttt{triv})[-p]$, whose intersection is the submodule generated by $v_{6c-3p}v_p$ isomorphic to $M_{1,c}(\texttt{sign})[-6c+2p]$.

□

12.2 The irreducible representation $L_{1,c}(\texttt{stand})$ characteristic $p>3$ for special c

The aim of this section is to prove Theorem 12.2.2 which describes $L_{1,c}(\texttt{stand})$ for $p>3$, $c\in\mathbb{F}_p$ with $0<c<p/3$. The proof of Theorem 12.2.2 relies on Lemma 12.2.17. We were unable to prove this lemma without a technical assumption on the parameter c. However, we believe the assumption is satisfied for all $c\in\mathbb{F}_p$ with $0<c<p/3$.

Throughout this section we will use $x^{\underline{n}}$ to denote to the falling factorial power

$$x^{\underline{n}} = \underbrace{x(x-1)(x-2)\cdots(x-n+1)}_{n \text{ factors}}$$

with the conventional empty product $x^{\underline{0}} = 1$.

Assumption 12.2.1. *Let $p > 3$ and $c \in \mathbb{F}_p$ with $0 < c < p/3$. Assume that c satisfies one of the following:*

1. *Either $0 < c < p/6$;*
2. *Or $p \equiv 1 \pmod 3$, $p/6 < c < p/3$, and*
$$\sum_{k=0}^{\lfloor \frac{1}{2}(\frac{p-1}{3}-c) \rfloor} \frac{1}{3^k} \frac{\prod_{j=1}^{k}(3j-1)(\frac{p-1}{3}-c)^{\underline{2k}}}{k!(c-2)^{\underline{2k}}} \neq 0;$$
3. *Or $p \equiv 2 \pmod 3$, $c = \frac{p+1}{6}$;*
4. *Or $p \equiv 2 \pmod 3$, $(p+1)/6 < c < p/3$, and*
$$\sum_{k=0}^{\lfloor \frac{1}{2}(\frac{p-2}{3}-c) \rfloor} \frac{1}{3^{k-1}} \frac{\prod_{j=1}^{k+1}(3j-2)(\frac{p-2}{3}-c)^{\underline{2k}}}{k!(c-1)^{\underline{2k+2}}} \neq 0.$$

We have checked this assumption in [Magma] for all $p < 2022$. We rely on this assumption in the proof of Lemma 12.2.17.

Theorem 12.2.2. *The irreducible representation $L_{1,c}(\texttt{stand})$ of the rational Cherednik algebra $H_{1,c}(S_3, \mathfrak{h})$ over an algebraically closed field of characteristic $p > 3$ for $c \in \mathbb{F}_p$, $0 < c < p/3$, is the quotient of the Verma module $M_{1,c}(\texttt{stand})$ by a submodule $\langle v_{p-3c}, v_+, v_-, v_{p+3c} \rangle$, with the singular vectors v_{p-3c}, v_+, v_- and v_{p+3c} found in Lemma 12.2.6, Corollary 11.2.8 and Lemma 12.2.37. Its character and Hilbert polynomial are*

$$\chi_{L_{1,c}(\texttt{stand})}(z) = \chi_{S(\mathfrak{h}^*)}(z) \cdot ([\texttt{stand}] - [\texttt{triv}]z^{p-3c} - [\texttt{stand}]z^p - [\texttt{sign}]z^{p+3c} + 2[\texttt{sign}]z^{2p})$$
$$Hilb_{L_{1,c}(\texttt{stand})}(z) = \frac{2 - z^{p-3c} - 2z^p - z^{p+3c} + 2z^{2p}}{(1-z)^2}.$$

Proof. The vectors v_{p-3c}, v_+,v_- and v_{p+3c} are shown to be singular in Lemma 12.2.6, Corollary 11.2.8 and Lemma 12.2.37. Once we have these vectors, the aim is to calculate the Hilbert polynomials of $L_{1,c}(\texttt{stand})$ and $M_{1,c}(\texttt{stand}) / \langle v_{p-3c}, v_+, v_-, v_{p+3c} \rangle$ and show they are both equal to $\frac{2-z^{p-3c}-2z^p-z^{p+3c}+2z^{2p}}{(1-z)^2}$. The claim of the Theorem will then follow immediately.

To calculate these Hilbert polynomials, we proceed as follows. Any proper graded submodule of $M_{1,c}(\texttt{stand})$ is generated by singular vectors belonging to some irreducible subrepresentation of S_3. Therefore each proper graded submodule of $M_{1,c}(\texttt{stand})$ is, up to grading shift, $M_{1,c}(\texttt{stand})$, $M_{1,c}(\texttt{triv})$, $M_{1,c}(\texttt{sign})$, or some proper quotient of one of these Verma modules. Note that any proper quotient of a Verma module must itself be the quotient by a Verma module or a proper quotient of a Verma module, up to grading shift. The irreducible module $L_{1,c}(\texttt{stand})$ is the quotient of the Verma module $M_{1,c}(\texttt{stand})$ by a collection of proper graded submodules, therefore the Hilbert polynomial of $L_{1,c}(\texttt{stand})$ is a linear combination of the Hilbert series of $M_{1,c}(\texttt{stand})$, $M_{1,c}(\texttt{triv})$, $M_{1,c}(\texttt{sign})$, or their proper quotients, up to grading shift. Since we know that $L_{1,c}(\texttt{stand})$ is finite-dimensional, and that we can start this process with the finite-dimensional baby Verma module $N_{1,c}(\texttt{stand})$, it follows that this process ends in finitely many steps and results in a Hilbert polynomial has finitely many nonzero terms. Lemma 12.2.5 tells us that the module $M_{1,c}(\texttt{stand})$ is an extension of the modules $L_{1,c}(\texttt{stand})[-kp]$ with $k \geq 0$, $L_{1,c}(\texttt{triv})[-kp+3c]$ with $k \geq 1$ and $L_{1,c}(\texttt{sign})[-kp-3c]$ with $k \geq 0$. Consequently the Hilbert series of any quotient of $M_{1,c}(\texttt{stand})$ will be a linear combination, with integer coefficients, of terms of the form $\frac{2z^{kp}}{(1-z)^2}$, $\frac{z^{kp-3c}}{(1-z)^2}$ and $\frac{z^{kp+3c}}{(1-z)^2}$.

First, consider the degrees $0, 1, \ldots, p-1$ of $L_{1,c}(\texttt{stand})$. If there are no singular vectors in degrees $0, 1, \ldots, p-1$ of $M_{1,c}(\texttt{stand})$ then the Verma module and its irreducible quotient are identical in degrees $0, 1, \ldots p-1$. Lemma 12.2.5 tells us that the only degrees among $0, 1, \ldots, p-1$ where $M_{1,c}(\texttt{stand})$ can have singular vectors and thus differ from $L_{1,c}(\texttt{stand})$ are $p-3c$ and $3c$. Lemmas 12.2.6 and 12.2.17 look for those singular vectors, finding one in degree $p-3c$ and none in degree $3c$, and let us conclude that

$$\mathrm{Hilb}_{L_{1,c}(\texttt{stand})}(z) = \frac{1}{(1-z)^2}(2 - z^{p-3c} + O(z^p))$$

where $O(z^p)$ denotes any sum of terms with degrees greater than or equal to p.

This in turn lets us conclude that up to degree $p-1$, any quotient of $M_{1,c}(\texttt{stand})$ either looks exactly like $M_{1,c}(\texttt{stand})$ with no singular vectors in degrees $0, 1, \ldots, p-1$ removed and the Hilbert series $\frac{2}{(1-z)^2}$, or looks like $L_{1,c}(\texttt{stand})$ with the Hilbert series $\frac{2-z^{p-3c}+O(z^p)}{(1-z)^2}$.

Next, we turn our attention to $M_{1,c}(\texttt{stand})/\langle v_{p-3c}, v_+, v_-, v_{p+3c}\rangle$. Lemma 12.2.38 shows that the quotient $M_{1,c}(\texttt{stand})/\langle v_{p-3c}, v_{p+3c}\rangle$ has Hilbert series

$$\mathrm{Hilb}_{M_{1,c}(\texttt{stand})/\langle v_{p-3c}, v_{p+3c}\rangle}(z) = \frac{1}{(1-z)^2}(2 - z^{p-3c} - z^{p+3c}).$$

Lemma 12.2.40 then shows that v_+ and v_- are not zero in the quotient $M_{1,c}(\texttt{stand})/\langle v_{p-3c}, v_{p+3c}\rangle$. Thus, they generate a module $\langle v_+, v_-\rangle$ in degree p, which is a quotient of $M_{1,c}(\texttt{stand})$, and so, by the previous paragraph, up to degree $2p-1$ looks either like $M_{1,c}(\texttt{stand})[-p]$ with the Hilbert series $\frac{2z^p+O(z^{2p})}{(1-z)^2}$ or like $L_{1,c}(\texttt{stand})[-p]$ with the Hilbert series $\frac{2z^p-z^{2p-3c}+O(z^{2p})}{(1-z)^2}$. To see what the submodule $\langle v_+, v_-\rangle$ of $M_{1,c}(\texttt{stand})/\langle v_{p-3c,p+3c}\rangle$ looks like in degrees up to $2p-1$, we notice it is the image of the map

$$\varphi : M_{1,c}(\texttt{stand})[-p] \to M_{1,c}(\texttt{stand})/\langle v_{p-3c}, v_{p+3c}\rangle$$

defined by $\varphi(b_\pm)=v_\pm$. In Lemma 12.2.41 we show $\varphi(v_{p-3c})\neq 0$, which lets us conclude that $\langle v_+,v_-\rangle$ has the Hilbert series $\frac{2z^p+O(z^{2p})}{(1-z)^2}$ and that $M_{1,c}(\texttt{stand})/\langle v_{p-3c},v_+,v_-,v_{p+3c}\rangle$ has the Hilbert series

$$\mathrm{Hilb}_{M_{1,c}(\texttt{stand})/\langle v_{p-3c},v_+,v_-,v_{p+3c}\rangle}(z)=\frac{1}{(1-z)^2}\left(2-z^{p-3c}-2\cdot z^p-z^{p+3c}+O(z^{2p})\right). \tag{12.2.3}$$

Noting that degree $2p-1$ of that expression is 0 (see Lemma 12.2.42), we conclude that

$$M_{1,c}(\texttt{stand})/\langle v_{p-3c},v_+,v_-,v_{p+3c}\rangle$$

is finite-dimensional, concentrated in degrees up to $2p-2$, with the Hilbert series

$$\mathrm{Hilb}_{M_{1,c}(\texttt{stand})/\langle v_{p-3c},v_+,v_-,v_{p+3c}\rangle}(z)=\frac{1}{(1-z)^2}\left(2-z^{p-3c}-2\cdot z^p-z^{p+3c}+2z^{2p}\right).$$

(The last statement is clear because both are polynomials of degree $2p-2$, whose coefficients match up to degree $2p-1$).

We now return to $L_{1,c}(\texttt{stand})$. It is a quotient of $M_{1,c}(\texttt{stand})/\langle v_{p-3c},v_+v_-,v_{p+3c}\rangle$ by some submodule. This submodule is concentrated in degrees p and above by Lemmas 12.2.5, 12.2.6 and 12.2.17. Let M be the number of irreducible composition factors of this submodule of type $L_{1,c}(\texttt{stand})[-p]$, let N be the number of irreducible composition factors of this submodule of type $L_{1,c}(\texttt{triv})[-2p+3c]$ and let K be the number of irreducible composition factors of this submodule of type $L_{1,c}(\texttt{sign})[-p-3c]$. We have $M,K,N\geq 0$. In the Grothendieck group we can write

$$\begin{aligned}[L_{1,c}(\texttt{stand})]=&[M_{1,c}(\texttt{stand})/\langle v_{p-3c},v_+,v_-,v_{p+3c}\rangle]-\\&-M\cdot[L_{1,c}(\texttt{stand})[-p]]-K\cdot[L_{1,c}(\texttt{sign})[-p-3c]]-N\cdot[L_{1,c}(\texttt{triv})[-2p+3c]].\end{aligned}$$

In terms of Hilbert polynomials, using the previously calculated Hilbert polynomials of $M_{1,c}(\texttt{stand})/\langle v_{p-3c},v_+,v_-,v_{p+3c}\rangle$, $L_{1,c}(\texttt{triv})$ and $L_{1,c}(\texttt{sign})$, we have

$$\begin{aligned}\mathrm{Hilb}_{L_{1,c}(\texttt{stand})}(z)=&\mathrm{Hilb}_{M_{1,c}(\texttt{stand})/\langle v_{p-3c},v_+,v_-,v_{p+3c}\rangle}(z)-M\cdot z^p\cdot\mathrm{Hilb}_{L_{1,c}(\texttt{stand})}(z)-\\&-K\cdot z^{3c+p}\cdot\mathrm{Hilb}_{L_{1,c}(\texttt{sign})}(z)-N\cdot z^{2p-3c}\mathrm{Hilb}_{L_{1,c}(\texttt{triv})}(z).\end{aligned} \tag{12.2.4}$$

From here, Lemma 12.2.43 lets us conclude that $M=K=N=0$. This implies that

$$L_{1,c}(\texttt{stand})=M_{1,c}(\texttt{stand})/\langle v_{p-3c},v_+,v_-,v_{p+3c}\rangle,$$

and its Hilbert polynomial is as claimed. □

We will now state and prove all the lemmas used in the above proof.

Recall Lemma 10.0.2, which says that the action $\Omega|_\tau$ of the Casimir element on any irreducible representation τ consisting of singular vectors is by the scalars

$$\Omega|_{\texttt{triv}}=0,\quad \Omega|_{\texttt{sign}}=6c,\quad \Omega|_{\texttt{stand}}=3c.$$

This limits the degrees in which singular vectors can occur, because if τ is an irreducible representation consisting of singular vectors inside $M^k_{1,c}(\sigma)$ or its quotients, then $\Omega|_\tau = \Omega|_\sigma + k \in \mathbb{F}_p$. Thus, to describe all degrees k for which singular vectors of type τ can occur in $M^k_{1,c}(\texttt{stand})$ or its quotients, we need to describe all $k\in\mathbb{N}$ of the form $k=\Omega|_\tau-\Omega|_{\texttt{stand}}$. Their relative order is also important, as we look for singular vectors in order from smaller degrees to bigger, taking quotients each time and subsequently looking for singular vectors in the quotient. Putting that information together for $0<c<p/3$, we get the following lemma.

Lemma 12.2.5. *If τ is a an irreducible representation consisting of singular vectors inside $M^k_{1,c}(\texttt{stand})$ or its quotient, then the pairs (k,τ) are of the form:*

1. *For $c\in\{0,1,\dots p-1\}$ with $0<c<p/6$:*

$$\begin{array}{lccccccccccc} k: & 3c & < & p-3c & < & p & < & 3c+p & < & 2p-3c & < & \dots \\ \tau: & \texttt{sign} & & \texttt{triv} & & \texttt{stand} & & \texttt{sign} & & \texttt{triv} & & \end{array}$$

2. *For $c\in\{0,1,\dots p-1\}$ with $p/6<c<p/3$:*

$$\begin{array}{lccccccccccc} k: & p-3c & < & 3c & < & p & < & 2p-3c & < & 3c+p & < & \dots \\ \tau: & \texttt{triv} & & \texttt{sign} & & \texttt{stand} & & \texttt{triv} & & \texttt{sign} & & \end{array}$$

We will first examine the `triv` and `sign` isotypic components in degrees $p-3c$ and $3c$. The order matters, because some vectors in higher degrees might turn out to only be singular modulo lower degrees. It will turn out that there is a singular vector in this `triv` but not in this `sign` isotypic component.

Lemma 12.2.6. *There is a 1-dimensional space of singular vectors in degree $p-3c$ of the module $M_{1,c}(\texttt{stand})$, in the `triv` isotypic component, spanned by the v_{p-3c}, given by the following formulas:*

1. *If $p\equiv 1 \pmod 3$,*

$$\begin{aligned} v_{p-3c} = & \sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c)\rfloor} \frac{(-1)^i}{9^i}\frac{(\frac{p-1}{3}-c)^{\underline{2i}}}{i!\cdot\prod_{j=1}^{i}(3j-2)}\sigma_2^{3i}\sigma_3^{\frac{p-1}{3}-c-2i}\cdot(b_+\otimes b_+ + 3b_-\otimes b_-) \\ & + \sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c-1)\rfloor} \frac{(-1)^{i+1}}{6\cdot 9^i}\frac{(\frac{p-1}{3}-c)^{\underline{2i+1}}}{i!\cdot\prod_{j=1}^{i+1}(3j-2)}\sigma_2^{3i+1}\sigma_3^{\frac{p-1}{3}-c-2i-1}\cdot\Big((-b_+^2+3b_-^2)\otimes b_+ \\ & \qquad + 3\cdot(2b_+b_-)\otimes b_-\Big); \end{aligned}$$

2. *If* $p \equiv 2 \pmod 3$,

$$v_{p-3c} = \sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-2}{3}-c-1)\rfloor} \frac{(-1)^i}{9^i}\frac{2(\frac{p-2}{3}-c)^{\underline{2i+1}}}{i!\prod_{j=1}^{i+1}(3j-1)}\sigma_2^{3i+2}\sigma_3^{\frac{p-2}{3}-c-2i-1}\cdot(b_+\otimes b_+ + 3b_-\otimes b_-)$$
$$+\sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-2}{3}-c)\rfloor} \frac{(-1)^i}{9^i}\frac{(\frac{p-2}{3}-c)^{\underline{2i}}}{i!\prod_{j=1}^{i}(3j-1)}\sigma_2^{3i}\sigma_3^{\frac{p-2}{3}-c-2i}\cdot\left((-b_+^2+3b_-^2)\otimes b_+ + 3\cdot(2b_+b_-)\otimes b_-\right).$$

Proof. According to Theorem 7.2.4, a basis for the `triv` isotypic component of $M_{1,c}^{p-3c}(\texttt{stand})$ is

$$\sigma_2^a\sigma_3^b\cdot(b_+\otimes b_+ + 3b_-\otimes b_-),\quad 2a+3b=p-3c-1$$
$$\sigma_2^a\sigma_3^b\cdot\left((-b_+^2+3b_-^2)\otimes b_+ + 3\cdot(2b_+b_-)\otimes b_-\right),\quad 2a+3b=p-3c-2.$$

We calculate the action of the Dunkl operator D_{y_1} on such vectors of total degree $p-3c$, using the auxiliary computations from Chapter 10, to be:

$$\begin{aligned}
&D_{y_1}(\sigma_2^a\sigma_3^b\cdot(b_+\otimes b_+ + 3b_-\otimes b_-)) = \qquad (12.2.7)\\
&=\frac{3c+1}{2}\sigma_2^a\sigma_3^b\otimes(b_+ + 3b_-)\\
&+\frac{b}{6}\sigma_2^{a+1}\sigma_3^{b-1}\cdot(b_+\otimes b_+ - 3b_-\otimes b_+ - 3b_+\otimes b_- - 3b_-\otimes b_-)\\
&+\frac{a}{12}\sigma_2^{a-1}\sigma_3^b\cdot\left((-b_+^2+3b_-^2)\otimes b_+ - 3(2b_+b_-)\otimes b_+ - 3(-b_+^2+3b_-^2)\otimes b_- - 3(2b_+b_-)\otimes b_-\right)\\
&+\frac{b}{2}\sigma_2^a\sigma_3^{b-1}q\otimes(b_+ - b_-);
\end{aligned}$$

$$\begin{aligned}
&D_{y_1}(\sigma_2^a\sigma_3^b\cdot\left((-b_+^2+3b_-^2)\otimes b_+ + 3\cdot(2b_+b_-)\otimes b_-\right)) = \qquad (12.2.8)\\
&=-9a\sigma_2^{a-1}\sigma_3^{b+1}\otimes(b_+ + 3b_-) + 2b\sigma_2^{a+2}\sigma_3^{b-1}\otimes(b_+ + 3b_-)\\
&+a\sigma_2^a\sigma_3^b\cdot(b_+\otimes b_+ - 3b_-\otimes b_+ - 3b_+\otimes b_- - 3b_-\otimes b_-)\\
&+\frac{-b}{6}\sigma_2^{a+1}\sigma_3^{b-1}\cdot\left((-b_+^2+3b_-^2)\otimes b_+ - 3(2b_+b_-)\otimes b_+ - 3(-b_+^2+3b_-^2)\otimes b_- - 3(2b_+b_-)\otimes b_-\right)\\
&+3a\sigma_2^{a-1}\sigma_3^b q\otimes(b_+ - b_-).
\end{aligned}$$

We now distinguish the cases $p\equiv 1 \pmod 3$ and $p\equiv 2\pmod 3$, in order to parametrize the integral solutions to $2a+3b=p-3c-1$ and $2a+3b=p-3c-2$ differently.

1. Assume $p\equiv 1\pmod 3$. Any vector in the `triv` isotypic component of $M_{1,c}^{p-3c}(\texttt{stand})$ is of the form

$$v_{p-3c} = \sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c)\rfloor}\lambda_i\sigma_2^{3i}\sigma_3^{\frac{p-1}{3}-c-2i}\cdot(b_+\otimes b_+ + 3b_-\otimes b_-)+$$

$$+\sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c-1)\rfloor}\mu_i\sigma_2^{3i+1}\sigma_3^{\frac{p-1}{3}-c-2i-1}\cdot\left((-b_+^2+3b_-^2)\otimes b_+ + 3\cdot(2b_+b_-)\otimes b_-\right).$$

We use equations (12.2.7) and (12.2.8) to express the condition $D_{y_1}(v_{p-3c})=0$ in terms of equations for the coefficients λ_i, μ_i.

First of all, reading off the coefficient of $q\otimes(b_+-b_-)$ we get that for all i

$$\lambda_i\cdot\frac{1}{2}(\frac{p-1}{3}-c-2i)+\mu_i\cdot 3(3i+1)=0. \tag{12.2.9}$$

The same condition gives that the coefficient of $\left(b_+\otimes b_+-3b_-\otimes b_+-3b_+\otimes b_--3b_-\otimes b_-\right)$ in $D_{y_1}(v_{p-3c})$ is 0.

Reading off the coefficient of $(-b_+^2+3b_-^2)\otimes b_+-3(2b_+b_-)\otimes b_+-3(-b_+^2+3b_-^2)\otimes b_- - 3(2b_+b_-)\otimes b_-$ we get that

$$\sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c)\rfloor}\lambda_i\frac{3i}{12}\sigma_2^{3i-1}\sigma_3^{\frac{p-1}{3}-c-2i}+\sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c-1)\rfloor}\mu_i\frac{-(\frac{p-1}{3}-c-2i-1)}{6}\sigma_2^{3i+2}\sigma_3^{\frac{p-1}{3}-c-2i-2}=0,$$

which can be rewritten as

$$\sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c)\rfloor-1}\lambda_{i+1}\frac{3(i+1)}{12}\sigma_2^{3i+2}\sigma_3^{\frac{p-1}{3}-c-2i-2}$$
$$+\sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c-1)\rfloor}\mu_i\frac{-(\frac{p-1}{3}-c-2i-1)}{6}\sigma_2^{3i+2}\sigma_3^{\frac{p-1}{3}-c-2i-2}=0$$

and leads to the condition that for all i

$$\lambda_{i+1}\frac{3(i+1)}{12}+\mu_i\frac{-(\frac{p-1}{3}-c-2i-1)}{6}=0 \tag{12.2.10}$$

Finally, reading off the coefficient of $\otimes(b_++3b_-)$ we get

$$\sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c)\rfloor}\lambda_i\frac{3c+1}{2}\sigma_2^{3i}\sigma_3^{\frac{p-1}{3}-c-2i}+\sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c-1)\rfloor}\mu_i(-9)(3i+1)\sigma_2^{3i}\sigma_3^{\frac{p-1}{3}-c-2i}$$
$$+\sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c-1)\rfloor}\mu_i\cdot 2\left(\frac{p-1}{3}-c-2i-1\right)\sigma_2^{3i+3}\sigma_3^{\frac{p-1}{3}-c-2i-2}=0.$$

This can be rewritten as

$$\sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c)\rfloor}\lambda_i\frac{3c+1}{2}\sigma_2^{3i}\sigma_3^{\frac{p-1}{3}-c-2i}+\sum_{i=0}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c-1)\rfloor}\mu_i(-9)(3i+1)\sigma_2^{3i}\sigma_3^{\frac{p-1}{3}-c-2i}$$
$$+\sum_{i=1}^{\lfloor\frac{1}{2}(\frac{p-1}{3}-c-1)\rfloor+1}\mu_{i-1}\cdot 2\left(\frac{p-1}{3}-c-2i+1\right)\sigma_2^{3i}\sigma_3^{\frac{p-1}{3}-c-2i}=0$$

and leads to

$$\lambda_i\frac{3c+1}{2}+\mu_i(-9)(3i+1)+\mu_{i-1}\cdot 2\left(\frac{p-1}{3}-c-2i+1\right)=0. \tag{12.2.11}$$

The system (12.2.9), (12.2.10) and (12.2.11) can be solved as follows. We first rewrite (12.2.9) as

$$\mu_i=\frac{-(\frac{p-1}{3}-c-2i)}{6(3i+1)}\cdot\lambda_i,$$

and then (12.2.10) as

$$\lambda_{i+1}=\frac{-(\frac{p-1}{3}-c-2i)(\frac{p-1}{3}-c-2i-1)}{9(3i+1)(i+1)}\cdot\lambda_i.$$

Choosing $\lambda_0=1$ (this is the overall scalar choice) and noting that $i+1\neq 0, 3i+1\neq 0$ for $0\leq i<\lfloor\frac{1}{2}(\frac{p-1}{3}-c)\rfloor$, we get that a solution to (12.2.9) and (12.2.10) is given by

$$\begin{aligned}\lambda_i&=\frac{(-1)^i}{9^i}\frac{\prod_{j=1}^{2i}(\frac{p-1}{3}-c+1-j)}{\prod_{j=1}^{i}(3j-2)\prod_{j=1}^{i}j}\\ \mu_i&=\frac{(-1)^{i+1}}{6\cdot 9^i}\frac{\prod_{j=1}^{2i+1}(\frac{p-1}{3}-c+1-j)}{\prod_{j=1}^{i+1}(3j-2)\prod_{j=1}^{i}j}.\end{aligned} \tag{12.2.12}$$

We note that any solution to (12.2.9) and (12.2.10) will also satisfy (12.2.11), as

$$\begin{aligned}&\lambda_i\frac{3c+1}{2}+\mu_i(-9)(3i+1)+\mu_{i-1}\cdot 2\left(\frac{p-1}{3}-c-2i+1\right)=\\ &=\lambda_i\frac{3c+1}{2}+(-9)(3i+1)\cdot\frac{-(\frac{p-1}{3}-c-2i)}{6(3i+1)}\cdot\lambda_i\\ &\quad+2\left(\frac{p-1}{3}-c-2i+1\right)\cdot\frac{3i}{2(\frac{p-1}{3}-c-2i+1)}\lambda i\\ &=\frac{\lambda_i}{2}\left((3c+1)+3\cdot\left(\frac{p-1}{3}-c-2i\right)\cdot+2\cdot 3i\right)=0.\end{aligned}$$

2. Assume $p\equiv 2\,(\mathrm{mod}\,3)$. Any vector in the `triv` isotypic component of $M_{1,c}^{p-3c}(\texttt{stand})$

is of the form

$$v_{p-3c} = \sum_{i=0}^{\lfloor \frac{1}{2}(\frac{p-2}{3}-c-1)\rfloor} \lambda_i \sigma_2^{3i+2} \sigma_3^{\frac{p-2}{3}-c-2i-1} \cdot (b_+ \otimes b_+ + 3b_- \otimes b_-) +$$
$$+ \sum_{i=0}^{\lfloor \frac{1}{2}(\frac{p-2}{3}-c)\rfloor} \mu_i \sigma_2^{3i} \sigma_3^{\frac{p-2}{3}-c-2i} \cdot \left((-b_+^2 + 3b_-^2) \otimes b_+ + 3 \cdot (2b_+b_-) \otimes b_-\right).$$

Using equations (12.2.7) and (12.2.8), the condition $D_{y_1}(v_{p-3c}) = 0$ leads to equations

$$\lambda_{i-1} \cdot \frac{1}{2}\left(\frac{p-2}{3} - c - 2i + 1\right) + \mu_i \cdot 3(3i) = 0 \qquad (12.2.13)$$
$$\lambda_i \cdot \frac{3i+2}{2} - \mu_i \cdot \left(\frac{p-1}{3} - c - 2i\right) = 0 \qquad (12.2.14)$$
$$\lambda_i \cdot \frac{3c+1}{2} - \mu_{i+1} \cdot 27(i+1) + \mu_i \cdot 2\left(\frac{p-2}{3} - c - 2i\right) = 0, \qquad (12.2.15)$$

which have a unique (up to overall scaling) solution

$$\lambda_i = \frac{(-1)^i}{9^i} \frac{2(\frac{p-1}{3} - c)^{\underline{2i+1}}}{i! \prod_{j=1}^{i+1}(3j-1)} \qquad (12.2.16)$$
$$\mu_i = \frac{(-1)^i}{9^i} \frac{(\frac{p-1}{3} - c)^{\underline{2i}}}{i! \prod_{j=1}^{i}(3j-1)}.$$

□

Lemma 12.2.17. *Suppose that Assumption 12.2.1 holds. There are no singular vectors in degree $3c$ of the module $M_{1,c}(\texttt{stand})/\langle v_{p-3c}\rangle$, where v_{p-3c} is the vector from Lemma 12.2.6.*

Proof. We distinguish the cases $0 < c < p/6$ and $p/6 < c < p/3$.

1. By Lemma 12.2.5, if $0 < c < p/6$ then $3c < p - 3c$ so degree $3c$ of the module $M_{1,c}(\texttt{stand})/\langle v_{p-3c}\rangle$ is equal to degree $3c$ of the module $M_{1,c}(\texttt{stand})$, and it is enough to show there are no singular vectors in the $\texttt{sign}$ component of $M_{1,c}^{3c}(\texttt{stand})$. This is the easier case as we are not working modulo v_{p-3c}. By Theorem 7.2.4, any vector in the $\texttt{sign}$ component of $M_{1,c}^{3c}(\texttt{stand})$ can be uniquely expressed as

$$v = \sum_{i=0}^{\lfloor \frac{1}{2}(c-1)\rfloor} \nu_i \sigma_2^{3i+1} \sigma_3^{c-1-2i} \cdot (b_+ \otimes b_- - b_- \otimes b_+)$$
$$+ \sum_{i=0}^{\lfloor \frac{1}{2}(c-2)\rfloor} \xi_i \sigma_2^{3i+2} \sigma_3^{c-2-2i} \cdot \left((-b_+^2 + 3b_-^2) \otimes b_- - (2b_+b_+) \otimes b_+\right).$$

We calculate that

$$\begin{aligned}
&D_{y_1}(\sigma_2^a\sigma_3^b\cdot(b_+\otimes b_- - b_-\otimes b_+)) = \\
&\quad= \frac{1-3c}{2}\cdot\sigma_2^a\sigma_3^b\otimes(b_- - b_+)\\
&\quad+\frac{b}{6}\cdot\sigma_2^{a+1}\sigma_3^{b-1}\cdot(b_+\otimes b_+ + b_-\otimes b_+ + b_+\otimes b_- - 3b_-\otimes b_-))\\
&\quad+\frac{a}{12}\cdot\sigma_2^{a-1}\sigma_3^b\cdot\Big((-b_+^2+3b_-^2)\otimes b_+ + (2b_+b_-)\otimes b_+\\
&\qquad\qquad + (-b_+^2+3b_-^2)\otimes b_- - 3(2b_+b_-)\otimes b_-)\Big)\\
&\quad+\frac{b}{6}\cdot\sigma_2^a\sigma_3^{b-1}q\otimes(3b_- - b_+),
\end{aligned}\tag{12.2.18}$$

$$\begin{aligned}
&D_{y_1}(\sigma_2^a\sigma_3^b\cdot\big((-b_+^2+3b_-^2)\otimes b_- - (2b_+b_+)\otimes b_-\big)) = \\
&\quad= -9a\cdot\sigma_2^{a-1}\sigma_3^{b+1}\otimes(b_- - b_+) + 2b\sigma_2^{a+2}\sigma_3^{b-1}\otimes(b_- - b_+)\\
&\quad+a\cdot\sigma_2^a\sigma_3^b\cdot(b_+\otimes b_+ + b_-\otimes b_+ + b_+\otimes b_- - 3b_-\otimes b_-))\\
&\quad-\frac{b}{6}\cdot\sigma_2^{a+1}\sigma_3^{b-1}\cdot\Big((-b_+^2+3b_-^2)\otimes b_+ + (2b_+b_-)\otimes b_+\\
&\qquad\qquad + (-b_+^2+3b_-^2)\otimes b_- - 3(2b_+b_-)\otimes b_-)\Big)\\
&\quad+a\cdot\sigma_2^{a-1}\sigma_3^b q\otimes(3b_- - b_+).
\end{aligned}\tag{12.2.19}$$

From here, $D_{y_1}(v)=0$ gives the system

$$\begin{aligned}
\frac{1-3c}{2}\cdot\nu_i - 9(3i+2)\cdot\xi_i + 2(c-2i)\cdot\xi_{i-1} &= 0\\
\frac{c-1-2i}{6}\cdot\nu_i + (3i+2)\cdot\xi_i &= 0\\
\frac{3i+1}{12}\cdot\nu_i - \frac{c-2i}{6}\cdot\xi_{i-1} &= 0,
\end{aligned}$$

the only solution to which is $\nu_i=0,\xi_i=0$ for all i, so $v=0$. The first equation follows from the second and third, so nothing new is gained. The thing to notice is that in the second and third equation all coefficients are never 0, except $c-1-2i$ when c is the biggest possible $c=\frac{c-1}{2}$ and c odd. Considering $i=0$ in the third equation, we can conclude $\nu_0=0$, then inductively use the second equation to conclude that if $\nu_i=0$ then $\xi_i=0$, and the third equation again to conclude that if $\xi_i=0$ then $\nu_{i+1}=0$ as well, so all $\nu_i=0,\xi_i=0$ for all i.

2. If $p/6<c<p/3$, the task is harder because $3c>p-3c$ so degree $3c$ of the module $M_{1,c}(\texttt{stand})/\langle v_{p-3c}\rangle$ is not equal to degree $3c$ of the module $M_{1,c}(\texttt{stand})$, and we have

to work modulo v_{p-3c}. We again distinguish two cases.

(a) If $p \equiv 1 \pmod 3$, an arbitrary vector in the `sign` component of $M_{1,c}^{3c}(\mathtt{stand})/\langle v_{p-3c}\rangle$ can be uniquely written as

$$v = \sum_{i=0}^{\lfloor \frac{c-1}{2}\rfloor} \nu_i \sigma_2^{3i+1}\sigma_3^{c-1-2i}\cdot(b_+\otimes b_- - b_-\otimes b_+)$$
$$+ \sum_{i=0}^{\frac{p-1}{6}-\lfloor\frac{c-1}{2}\rfloor-2} \xi_i\sigma_2^{3i+2}\sigma_3^{c-2-2i}\cdot\big((-b_+^2+3b_-^2)\otimes b_- - (2b_+b_+)\otimes b_-\big).$$

Note the different boundaries of summation, reflecting the fact that in the quotient $M_{1,c}(\mathtt{stand})/\langle v_{p-3c}\rangle$ the multiplicity of `sign` in degree $3c$ is $\frac{p-1}{6}$, and the terms in the above summation form a basis.

Such a vector v is singular in the quotient $M_{1,c}^{3c}(\mathtt{stand})/\langle v_{p-3c}\rangle$ if and only if

$$D_{y_1}(v)\in\langle v_{p-3c}\rangle,$$

meaning that there exist $\pi_j,\rho_j\in\Bbbk$ such that

$$D_{y_1}(v) = \sum_{j=0}^{c-\frac{p-1}{6}-1}\pi_j\sigma_2^{3j}\sigma_3^{2c-\frac{p-1}{3}-1-2j}v_{p-3c}(-b_++b_-)$$
$$+\sum_{j=0}^{c-\frac{p-1}{6}-1}\rho_j\sigma_2^{3j+1}\sigma_3^{2c-\frac{p-1}{3}-2-2j}v_{p-3c}(-(-b_+^2+3b_-^2)+(2b_+b_-)).$$

Expanding this out using the explicit form for v_{p-3c} from Lemma 12.2.6 and equations (12.2.18) and (12.2.19) we get that this condition is equivalent to the folllowing system of equations being satisfied for all k:

$$\nu_k\cdot\frac{1-3c}{2}+\xi_k\cdot(-9)(3k+2)+\xi_{k-1}\cdot 2(c-2k)=$$
$$=\sum_{j=0}^{k}-6\pi_j\lambda_{k-j}+54\pi_j\mu_{k-j}+54\rho_j\lambda_{k-j}+72\rho_j\mu_{k-j} \tag{12.2.20}$$

$$\nu_k\cdot\frac{c-1-2k}{6}+\xi_k\cdot(3k+2)=\sum_{j=0}^{k}-6\pi_j\mu_{k-j}-6\rho_j\lambda_{k-j}+108\rho_j\mu_{k-j} \tag{12.2.21}$$

$$\nu_k\cdot\frac{3k+1}{12}+\xi_{k-1}\cdot\frac{-c+2k}{6}=\sum_{j=0}^{k}\frac{1}{2}\pi_j\lambda_{k-j}+6\rho_j\mu_{k-j-1} \tag{12.2.22}$$

$$\nu_k \cdot \frac{c-1-2k}{6} + \xi_k \cdot (3k+2) = \sum_{j=0}^{k} -6\pi_j\mu_{k-j} + 6\rho_j\lambda_{k-j}, \tag{12.2.23}$$

where

$$\begin{aligned} \lambda_i &= \frac{(-1)^i}{9^i} \frac{(\frac{p-1}{3}-c)^{\underline{2i}}}{i! \cdot \prod_{j=1}^{i}(3j-2)} \\ \mu_i &= \frac{(-1)^{i+1}}{6 \cdot 9^i} \frac{(\frac{p-1}{3}-c)^{\underline{2i+1}}}{i! \cdot \prod_{j=1}^{i+1}(3j-2)} \end{aligned} \tag{12.2.24}$$

as in Lemma 12.2.6.

Subtracting equations (12.2.21) and (12.2.23) we get that for all k

$$\sum_{j=0}^{k} \rho_j(-12\lambda_{k-j} + 108\mu_{k-j}) = 0,$$

which, using Equations (12.2.25) to check that $-12\lambda_{k-j} + 108\mu_{k-j} \neq 0$ implies that $\rho_j = 0$ for all j.

The system (12.2.20)-(12.2.23) is now equivalent to asking that for all k

$$\nu_k \cdot \frac{c-1-2k}{6} + \xi_k \cdot (3k+2) = -6\sum_{j=0}^{k} \pi_j\mu_{k-j} \tag{12.2.25}$$

$$\nu_k \cdot \frac{3k+1}{12} + \xi_{k-1} \cdot \frac{-c+2k}{6} = \frac{1}{2}\sum_{j=0}^{k} \pi_j\lambda_{k-j}. \tag{12.2.26}$$

We split into two further cases.

i. Assume c is even. The relevance of this assumption is in the range of k for which the unknowns ν_k, ξ_k, π_k and the parameters λ_k, μ_k are potentially nonzero. Namely, we are looking for all solutions to (12.2.25)-(12.2.26) where:

$$\begin{aligned} \nu_k &: \ 0 \le k \le \frac{c}{2} - 1 \\ \xi_k &: \ 0 \le k \le \frac{p-1}{6} - \frac{c}{2} - 1 \\ \pi_k &: \ 0 \le k \le c - \frac{p-1}{6} - 1 \\ \lambda_k &: \ 0 \le k \le \frac{p-1}{6} - \frac{c}{2} - 1 \\ \mu_k &: \ 0 \le k \le \frac{p-1}{6} - \frac{c}{2} - 1. \end{aligned}$$

Let us prove by downwards induction on m that

$$\nu_{\frac{p-1}{6}-\frac{c}{2}+m}=0 \quad \pi_m=0 \qquad \text{for all } m>0.$$

It is certainly true for large m (by convention). Assume it is true for $m+1, m+2, \ldots$, and let us prove it for m. Equation (12.2.25) for $k=\frac{p-1}{6}-\frac{c}{2}+m$ reads $\nu_{\frac{p-1}{6}-\frac{c}{2}+m}=0$, and equation (12.2.26) for $k=\frac{p-1}{6}-\frac{c}{2}+m$ implies $\pi_m=0$.

The system (12.2.25)-(12.2.26) is thus reduced to

$$\nu_k\cdot\frac{c-1-2k}{6}+\xi_k\cdot(3k+2)=-6\pi_0\mu_k \tag{12.2.27}$$

$$\nu_k\cdot\frac{3k+1}{12}+\xi_{k-1}\cdot\frac{-c+2k}{6}=\frac{1}{2}\pi_0\lambda_k. \tag{12.2.28}$$

We multiply (12.2.27) by a_k and (12.2.28) by b_k for

$$a_k=\frac{(-1)^{k+1}}{2}\frac{(3k+1)!}{k!(c-1)^{\underline{2k+1}}}, \quad b_k=(-1)^k\frac{(3k)!}{k!(c-1)^{\underline{2k}}},$$

and sum over all k to get,

$$\begin{aligned}
0&=\pi_0\cdot\left(\sum_{k=0}^{\frac{p-1}{6}-\frac{c}{2}-1}(-6)\mu_k a_k+\sum_{k=0}^{\frac{p-1}{6}-\frac{c}{2}}\frac{1}{2}\lambda_k b_k\right)\\
&=\pi_0\cdot\left(\sum_{k=0}^{\frac{p-1}{6}-\frac{c}{2}-1}\frac{-1}{2\cdot 3^k}\frac{\prod_{j=1}^k(3j-1)(\frac{p-1}{3}-c)^{\underline{2k+1}}}{k!(c-1)^{\underline{2k+1}}}\right.\\
&\qquad\left.+\sum_{k=0}^{\frac{p-1}{6}-\frac{c}{2}}\frac{1}{2\cdot 3^k}\frac{\prod_{j=1}^k(3j-1)(\frac{p-1}{3}-c)^{\underline{2k}}}{k!(c-1)^{\underline{2k}}}\right)\\
&=\pi_0\cdot\frac{c-\frac{p-1}{6}-\frac{1}{2}}{c-1}\sum_{k=0}^{\frac{p-1}{6}-\frac{c}{2}}\frac{1}{3^k}\frac{\prod_{j=1}^k(3j-1)(\frac{p-1}{3}-c)^{\underline{2k}}}{k!(c-2)^{\underline{2k}}}.
\end{aligned}$$

As $\frac{c-\frac{p-1}{6}-\frac{1}{2}}{c-1}\neq 0$ for $c\in\mathbb{F}_p$, $p/6<c<p/3$, and as

$$\sum_{k=0}^{\frac{p-1}{6}-\frac{c}{2}}\frac{1}{3^k}\frac{\prod_{j=1}^k(3j-1)(\frac{p-1}{3}-c)^{\underline{2k}}}{k!(c-2)^{\underline{2k}}}\neq 0$$

by Assumption 12.2.1, we conclude that $\pi_0=0$. From here we proceed like in part (1) to deduce that $\nu_k=0$, $\xi_k=0$ for all k, and $v=0$.

ii. Assume c is odd. In this case we are looking for all solutions to (12.2.25)-

(12.2.26) with the boundaries:

$$\begin{aligned}\nu_k:&\ 0\le k\le \frac{c-1}{2}\\ \xi_k:&\ 0\le k\le \frac{p-1}{6}-\frac{c-1}{2}-2\\ \pi_k:&\ 0\le k\le c-\frac{p-1}{6}-1\\ \lambda_k:&\ 0\le k\le \frac{p-1}{6}-\frac{c-1}{2}-1\\ \mu_k:&\ 0\le k\le \frac{p-1}{6}-\frac{c-1}{2}-1.\end{aligned}$$

As before, we prove by downwards induction on m that

$$\nu_{\frac{p-1}{6}-\frac{c-1}{2}-1+m}=0 \quad \pi_m=0 \quad \text{for all } m>0.$$

Assuming this holds for $m+1, m+2, \dots$, we write equations (12.2.25)-(12.2.26) for $k=\frac{p-1}{6}-\frac{c-1}{2}-1+m$ to get a system of two equations with two unknowns $\nu_{\frac{p-1}{6}-\frac{c-1}{2}-1+m}$ and $\pi_m=0$, the only solution of which is $\nu_{\frac{p-1}{6}-\frac{c-1}{2}-1+m}=0$, $\pi_m=0$.

The remaining system is identical to the system (12.2.27)-(12.2.28), and we proceed as in case (i).

(b) If $p\equiv 2 \pmod 3$, similar. An arbitrary vector in the $\texttt{sign}$ component of $M^{3c}(\texttt{stand})/\langle v_{p-3c}\rangle$ (which has dimension $\frac{p+1}{6}$) can be uniquely written as

$$\begin{aligned}v=&\sum_{i=0}^{\lfloor\frac{c-1}{2}\rfloor}\nu_i\sigma_2^{3i+1}\sigma_3^{c-1-2i}\cdot(b_+\otimes b_- - b_-\otimes b_+)\\ &+\sum_{i=0}^{\frac{p+1}{6}-\lfloor\frac{c-1}{2}\rfloor-2}\xi_i\sigma_2^{3i+2}\sigma_3^{c-2-2i}\cdot\big((-b_+^2+3b_-^2)\otimes b_- - (2b_+b_+)\otimes b_-\big)\big).\end{aligned}$$

Such a vector is singular in $M(\texttt{stand})/\langle v_{p-3c}\rangle$ if and only if there exist $\pi_j,\rho_j\in\Bbbk$ such that

$$\begin{aligned}D_{y_1}(v)=&\sum_{j=0}^{c-\frac{p+1}{6}-1}\pi_j\sigma_2^{3j+1}\sigma_3^{2c-\frac{p+1}{3}-2j-1}v_{p-3c}(-b_++b_-)\\ &+\sum_{j=0}^{c-\frac{p+1}{6}-1}\rho_j\sigma_2^{3j+2}\sigma_3^{2c-\frac{p+1}{3}-2-2j}v_{p-3c}(-(-b_+^2+3b_-^2)+(2b_+b_-)).\end{aligned}$$

This is equivalent to the following system being satisfied for all k:

$$\nu_k\cdot\frac{1-3c}{2}+\xi_k\cdot(-9)(3k+2)+\xi_{k-1}\cdot 2(c-2k)=$$

$$= \sum_{j=0} -6\pi_j\lambda_{k-j-1} + 54\pi_j\mu_{k-j} + 54\rho_j\lambda_{k-j-1} + 72\rho_j\mu_{k-j-1} \tag{12.2.29}$$

$$\nu_k \cdot \frac{c-1-2k}{6} + \xi_k \cdot (3k+2) = \sum_{j=0} -6\pi_j\mu_{k-j} - 6\rho_j\lambda_{k-j-1} + 108\rho_j\mu_{k-j} \tag{12.2.30}$$

$$\nu_k \cdot \frac{3k+1}{12} + \xi_{k-1} \cdot \frac{-c+2k}{6} = \sum_{j=0} \frac{1}{2}\pi_j\lambda_{k-j-1} + 6\rho_j\mu_{k-j-1} \tag{12.2.31}$$

$$\nu_k \cdot \frac{c-1-2k}{6} + \xi_k \cdot (3k+2) = \sum_{j=0} -6\pi_j\mu_{k-j} + 6\rho_j\lambda_{k-j-1}, \tag{12.2.32}$$

where λ_i, μ_i are as in Lemma 12.2.6.

Again, comparing equations (12.2.30) and (12.2.34) lets us show that that $\rho_j = 0$ for all j. Removing equations which are duplicated or linear combinations of other equations we get:

$$\nu_k \cdot \frac{c-1-2k}{6} + \xi_k \cdot (3k+2) = -6\sum_{j=0}^{k} \pi_j\mu_{k-j} \tag{12.2.33}$$

$$\nu_k \cdot \frac{3k+1}{12} + \xi_{k-1} \cdot \frac{-c+2k}{6} = \frac{1}{2}\sum_{j=0}^{k-1} \pi_j\lambda_{k-j-1}. \tag{12.2.34}$$

As before, we show by downwards induction on m that for all $m > 0$ we have $\pi_m = 0$ and $\nu_{\frac{p+1}{6}-\frac{c}{2}-1+m} = 0$ (if c even) or $\nu_{\frac{p+1}{6}-\frac{c-1}{2}-1+m} = 0$ (if c odd).

This leaves us with the system

$$\nu_k \cdot \frac{c-1-2k}{6} + \xi_k \cdot (3k+2) = -6\pi_0\mu_k \tag{12.2.35}$$

$$\nu_k \cdot \frac{3k+1}{12} + \xi_{k-1} \cdot \frac{-c+2k}{6} = \frac{1}{2}\pi_0\lambda_{k-1}. \tag{12.2.36}$$

When $c = \frac{p+1}{6}$, some of the coefficients in this system are 0, so we can not do telescoping like in the case $p \equiv 1 \pmod 3$. Instead, we notice that certain equations immediately give us $\pi_0 = 0$:

i. If c is odd, equation (12.2.35) for $k = \frac{p-5}{12}$ implies

$$\nu_k \cdot 0 + 0 \cdot (3k+2) = -6\pi_0\mu_k,$$

which, using $\mu_{\frac{p-5}{12}} \neq 0$, gives $\pi_0 = 0$.

ii. If c is even, equation (12.2.36) for $k = \frac{p-11}{12} + 1$ implies

$$0 + \xi_{\frac{p-11}{12}} \cdot 0 = \frac{1}{2}\pi_0 \lambda_{\frac{p-11}{12}},$$

which, using $\lambda_{\frac{p-11}{12}} \neq 0$, gives $\pi_0 = 0$.

If $c \neq \frac{p+1}{6}$, we use telescoping to get

$$0 = \pi_0 \cdot \sum_{k=0}^{\lfloor \frac{1}{2}(\frac{p-2}{3}-c)\rfloor} \frac{1}{3^{k-1}} \frac{\prod_{j=1}^{k+1}(3j-2)(\frac{p-2}{3}-c)^{\underline{2k}}}{k!(c-1)^{\underline{2k+2}}}.$$

Assumption 12.2.1 now implies $\pi_0 = 0$.

So, in all cases $\pi_0 = 0$, and we proceed like in part (1) to deduce that $\nu_k = 0$, $\xi_k = 0$ for all k, and $v = 0$.

□

Next, we search the degree $p+3c$ of $M_{1,c}(\texttt{stand})$ for singular vectors of type $\texttt{sign}$.

Lemma 12.2.37. *There is a 1-dimensional space of singular vectors in degree $p+3c$ of the module $M_{1,c}(\texttt{stand})$, in the $\texttt{sign}$ isotypic component, spanned by v_{p+3c}, which is given by the following formulas:*

1. *If $p \equiv 1 \pmod 3$,*

$$v_{p+3c} = \sum_{i=0}^{\lfloor \frac{1}{2}(\frac{p-1}{3}+c)\rfloor} \frac{(-1)^i}{9^i} \frac{(\frac{p-1}{3}+c)^{\underline{2i}}}{i! \cdot \prod_{j=1}^{i}(3j-2)} \sigma_2^{3i} \sigma_3^{\frac{p-1}{3}+c-2i} \cdot (b_+ \otimes b_- - b_- \otimes b_+)$$
$$+ \sum_{i=0}^{\lfloor \frac{1}{2}(\frac{p-1}{3}+c-1)\rfloor} \frac{(-1)^{i+1}}{6 \cdot 9^i} \frac{(\frac{p-1}{3}+c)^{\underline{2i+1}}}{i! \cdot \prod_{j=1}^{i+1}(3j-2)} \sigma_2^{3i+1} \sigma_3^{\frac{p-1}{3}+c-2i-1} \cdot \Big((-b_+^2 + 3b_-^2) \otimes b_+$$
$$+ 3 \cdot (2b_+b_-) \otimes b_-\Big);$$

2. *If $p \equiv 2 \pmod 3$,*

$$v_{p+3c} = \sum_{i=0}^{\lfloor \frac{1}{2}(\frac{p-2}{3}+c-1)\rfloor} \frac{(-1)^i}{9^i} \frac{2(\frac{p-2}{3}+c)^{\underline{2i+1}}}{i! \prod_{j=1}^{i+1}(3j-1)} \sigma_2^{3i+2} \sigma_3^{\frac{p-2}{3}+c-2i-1} \cdot (b_+ \otimes b_- - b_- \otimes b_+) +$$
$$+ \sum_{i=0}^{\lfloor \frac{1}{2}(\frac{p-2}{3}+c)\rfloor} \frac{(-1)^i}{9^i} \frac{(\frac{p-2}{3}+c)^{\underline{2i}}}{i! \prod_{j=1}^{i}(3j-1)} \sigma_2^{3i} \sigma_3^{\frac{p-2}{3}+c-2i} \cdot \left((-b_+^2 + 3b_-^2) \otimes b_- - (2b_+b_-) \otimes b_+\right).$$

Proof. The proof, very similar to the proofs of Lemmas 12.2.6 and 12.2.17, is a direct computation using formulas (12.2.18) and (12.2.19), showing that $D_{y_1}(v_{p+3c}) = 0$. □

After this, as outlined in the proof of Theorem 12.2.2, we proceed to calculate the Hilbert polynomial of the quotient of $M_{1,c}(\texttt{stand})$ by $v_{p-3c}, v_+, v_-, v_{p+3c}$.

Lemma 12.2.38. *The Hilbert polynomial of the quotient* $M_{1,c}(\texttt{stand})/\left\langle v_{p-3c,v_{p+3c}}\right\rangle$ *equals*

$$Hilb_{M_{1,c}(\texttt{stand})/\langle v_{p-3c}, v_{p+3c}\rangle} = \frac{1}{(1-z)^2}(2 - z^{p-3c} - z^{p+3c}).$$

Proof. The vector v_{p-3c} generates a submodule isomorphic to $M_{1,c}(\texttt{triv})[-p+3c]$, the vector v_{p+3c} generates a submodule isomorphic to $M_{1,c}(\texttt{sign})[-p-3c]$, so we only need to calculate the Hilbert series of the module $\langle v_{p-3c}\rangle \cap \langle v_{p-3c}\rangle$. Write

$$v_{p-3c} = f \otimes b_+ + g \otimes b_-$$
$$v_{p+3c} = h \otimes b_+ + k \otimes b_-$$

with $f, g, h, k \in S(\mathfrak{h}^*)$. A vector is in $\langle v_{p-3c}\rangle \cap \langle v_{p-3c}\rangle$ if and only if it can be written as $Av_{p-3c} = Bv_{p+3c}$ for some $A, B \in S(\mathfrak{h}^*)$. This leads to the system

$$\begin{aligned} Af - Bh &= 0 \\ Ag - Bk &= 0, \end{aligned} \tag{12.2.39}$$

and to see if this has any nontrivial solutions, we will calculate its determinant $fk - gh$.

First, the facts that v_{p-3c} is S_3 invariant and that v_{p+3c} is S_3 anti-invariant lead to

$$\begin{aligned}
&(12).f = f, && (12).g = -g, && (12).h = -h, && (12).k = k\\
&(23).f = \frac{-f+g}{2}, && (23).g = \frac{3f+g}{2}, && (23).h = \frac{h-k}{2}, && (23).k = \frac{-3h-k}{2}\\
&(13).f = \frac{-f-g}{2}, && (23).g = \frac{-3f+g}{2}, && (23).h = \frac{h+k}{2}, && (23).k = \frac{3h-k}{2}.
\end{aligned}$$

Next, the fact that v_{p-3c} and v_{p+3c} are both singular leads to

$$\begin{aligned}
\partial_{y_1}(f) &= -c\frac{1}{x_1 - x_3}\frac{3f+g}{2}\\
\partial_{y_1}(g) &= -2c\frac{g}{x_1 - x_2} - c\frac{1}{x_1 - x_3}\frac{3f+g}{2}\\
\partial_{y_1}(h) &= c\frac{2h}{x_1 - x_2} + c\frac{1}{x_1 - x_3}\frac{h-k}{2}\\
\partial_{y_1}(k) &= -3c\frac{1}{x_1 - x_3}\frac{h-k}{2}.
\end{aligned}$$

From here it is straightforward to show that $fk - gh$ is an S_3 invariant with the property $D_{y_1}(fk - gh) = 0$, so $D_y(fk - gh) = 0$ for all y and thus $fk - gh$ is a p-th power and an invariant. Given that its degree is $2p$, we conclude that $fk - gh$ is a scalar multiple of σ_2^p.

To calculate this scalar, we calculate $fk - gh$ explicitly, using formulas in Lemmas 12.2.6

and 12.2.37, while disregarding all terms with a nonzero power of σ_3 (as we know from the above that they don't contribute to the final expression as $fk-gh$ is a scalar multiple of σ_2^p). We distinguish four cases, and claim that in all of them $fk-gh \neq 0$.

1. If $p \equiv 1 \pmod 3$ and $m = \frac{p-1}{3} - c$ is even, then $\frac{p-1}{3} + c = m + 2c$ is even and we get

$$\begin{aligned}
v_{p-3c} &= \frac{(-1)^{m/2}}{9^{m/2}} \frac{m!}{(m/2)! \cdot \prod_{j=1}^{m/2}(3j-2)} \sigma_2^{3m/2} \cdot (b_+ \otimes b_+ + 3b_- \otimes b_-) + \sigma_3 \dots \\
v_{p+3c} &= \frac{(-1)^{m/2+c}}{9^{m/2+c}} \frac{(m+2c)!}{(m/2+c)! \cdot \prod_{j=1}^{m/2+c}(3j-2)} \sigma_2^{3(m/2+c)} \cdot (b_+ \otimes b_- - b_- \otimes b_+) + \sigma_3 \dots \\
fk - gh &= \frac{(-1)^{m/2}}{9^{m/2}} \frac{m!}{(m/2)! \cdot \prod_{j=1}^{m/2}(3j-2)} \cdot \frac{(-1)^{m/2+c}}{9^{m/2+c}} \frac{(m+2c)!}{(m/2+c)! \cdot \prod_{j=1}^{m/2+c}(3j-2)} \cdot \\
&\qquad \cdot \sigma_2^{3m/2} \sigma_2^{3(m/2+c)} \cdot (b_+^2 + 3b_-^2) \\
&= \frac{(-1)^{\frac{p-1}{3}}}{9^{\frac{p-1}{3}}} \cdot \frac{(-12)(\frac{p-1}{3}-c)!(\frac{p-1}{3}+c)!}{(\frac{1}{2}(\frac{p-1}{3}-c))!(\frac{1}{2}(\frac{p-1}{3}+c))! \cdot \prod_{j=1}^{\frac{1}{2}(\frac{p-1}{3}-c)}(3j-2) \cdot \prod_{j=1}^{\frac{1}{2}(\frac{p-1}{3}+c)}(3j-2)} \cdot \sigma_2^p \\
&\neq 0.
\end{aligned}$$

2. If $p \equiv 1 \pmod 3$ and $m = \frac{p-1}{3} - c$ is odd a similar proof shows that $fk - gh \neq 0$.
3. If $p \equiv 2 \pmod 3$ and $m = \frac{p-2}{3} - c$ is even a similar proof shows that $fk - gh \neq 0$.
4. If $p \equiv 2 \pmod 3$ and $m = \frac{p-2}{3} - c$ is odd a similar proof shows that $fk - gh \neq 0$.

We showed that $fk-gh \neq 0$, which shows that the only solutions to the system (12.2.39) is $A = B = 0$, which means that $\langle v_{p-3c}\rangle \cap \langle v_{p-3c}\rangle = 0$. Consequently,

$$\begin{aligned}
\mathrm{Hilb}_{M_{1,c}(\mathtt{stand})/\langle v_{p-3c}, v_{p+3c}\rangle} &= \mathrm{Hilb}_{M_{1,c}(\mathtt{stand})} - \mathrm{Hilb}_{\langle v_{p-3c}\rangle} - \mathrm{Hilb}_{\langle v_{p+3c}\rangle} + \mathrm{Hilb}_{\langle v_{p-3c}\rangle \cap \langle v_{p-3c}\rangle} \\
&= \frac{2}{(1-z)^2} - z^{p-3c} \cdot \frac{1}{(1-z)^2} - z^{p+3c} \cdot \frac{1}{(1-z)^2} + 0.
\end{aligned}$$

□

Lemma 12.2.40. *The images of the vectors v_+, v_- from Lemma 11.2.8 in the quotient $M_{1,c}(\mathtt{stand})/\langle v_{p-3c}, v_{p+3c}\rangle$ are nonzero.*

Proof. Let us show that v_+ is not contained in $\langle v_{p-3c}, v_{p+3c}\rangle$. As v_- can be calculated from v_+ by the action of $\Bbbk(S_3) \subseteq H_{1,c}(S_3, \mathfrak{h})$ and $\langle v_{p-3c}, v_{p+3c}\rangle$ is an $H_{1,c}(S_3, \mathfrak{h})$ submodule, so the claim for v_- will follow without any further computations. Further, noting that $\deg v_+ = p > p - 3c = \deg v_{p-3c}$ and $\deg v_+ = p < p + 3c = \deg v_{p+3c}$, the claim is equivalent to showing that $v_+ \notin \langle v_{p-3c}\rangle$.

Assume the contrary, that $v_+ \in \langle v_{p-3c}\rangle$. In that case there exists $A \in S^{3c}(\mathfrak{h}^*)$ such that $Av_{p-3c} = v_+$. Considering how these vectors transform under the action of S_3 (v_{p-3c} spans triv and v_+ spans the (12)-invariant part of stand), we determine that A needs to transform like the (12)-invariant part of stand and thus needs to be of the form

$$A = gb_+ + h(-b_+^2 + 3b_-^2)$$

for some $g \in S^{3c-1}(\mathfrak{h}^*)^{S_3}$, $h \in S^{3c-2}(\mathfrak{h}^*)^{S_3}$. We will consider the equation $Av_{p-3c} = v_+$ modulo σ_2 to show it has no solutions. For that purpose, we distinguish two cases.

1. If $p \equiv 1 \pmod 3$, then

$$\begin{aligned} v_+ &= \sigma_3^{\frac{p-1}{3}}(-b_+ \otimes b_+ + 3b_- \otimes b_-) \bmod \sigma_2 \\ v_{p-3c} &= \sigma_3^{\frac{p-1}{3}-c}(b_+ \otimes b_+ + 3b_- \otimes b_-) \bmod \sigma_2. \end{aligned}$$

   Modulo σ_2, the equation $Av_{p-3c} = v_+$ becomes

$$\begin{aligned} \sigma_3^{\frac{p-1}{3}}(-b_+ \otimes b_+ + 3b_- \otimes b_-) &= (gb_+ + h(-b_+^2 + 3b_-^2)) \cdot \sigma_3^{\frac{p-1}{3}-c}(b_+ \otimes b_+ + 3b_- \otimes b_-) \bmod \sigma_2 \\ \sigma_3^c(-b_+ \otimes b_+ + 3b_- \otimes b_-) &= (gb_+^2 + h(-b_+^2 + 3b_-^2) \cdot b_+) \otimes b_+ \\ &\quad + (3gb_+b_- + 3h(-b_+^2 + 3b_-^2) \cdot b_-) \otimes b_- \bmod \sigma_2 \\ &= \left(\frac{-1}{2}g(-b_+^2 + 3b_-^2) + h \cdot 54\sigma_3\right) \otimes b_+ \\ &\quad + \left(\frac{1}{2}g(2b_+b_-) + 3h \cdot (-6q)\right) \otimes b_- \bmod \sigma_2. \end{aligned}$$

   Reading off the coefficient of $(-b_+ \otimes b_+ + 3b_- \otimes b_-)$ we get

$$\sigma_3^c = 0 \bmod \sigma_2$$

   which is a contradiction.

2. If $p \equiv 2 \pmod 3$, then

$$\begin{aligned} v_+ &= \sigma_3^{\frac{p-2}{3}}(-(-b_+^2 + 3b_-^2) \otimes b_+ + 3(2b_+b_-) \otimes b_-) \bmod \sigma_2 \\ v_{p-3c} &= \sigma_3^{\frac{p-2}{3}-c}((-b_+^2 + 3b_-^2) \otimes b_+ + 3(2b_+b_-) \otimes b_-) \bmod \sigma_2. \end{aligned}$$

   The equation $Av_{p-3c} = v_+$, similar to above, becomes

$$\begin{aligned} &\sigma_3^{\frac{p-2}{3}}(-(-b_+^2 + 3b_-^2) \otimes b_+ + 3(2b_+b_-) \otimes b_-) = \\ &\qquad = (gb_+ + h(-b_+^2 + 3b_-^2)) \cdot \sigma_3^{\frac{p-2}{3}-c}((-b_+^2 + 3b_-^2) \otimes b_+ + 3(2b_+b_-) \otimes b_-) \\ &\sigma_3^c(-(-b_+^2 + 3b_-^2) \otimes b_+ + 3(2b_+b_-) \otimes b_-) = \\ &\qquad = (gb_+(-b_+^2 + 3b_-^2) + h(-b_+^2 + 3b_-^2)^2) \otimes b_+ \end{aligned}$$

$$+\,3(gb_+(2b_+b_-)+h(-b_+^2+3b_-^2)(2b_+b_-))\otimes b_-$$

$$\sigma_3^c(-(-b_+^2+3b_-^2)\otimes b_+ + 3(2b_+b_-)\otimes b_-) = (g\cdot 54\sigma_3 + h\cdot(-108)\sigma_3 b_+)\otimes b_+ \\ +\,3(g\cdot 6q + h\cdot 108\sigma_3 b_-)\otimes b_- \bmod \sigma_2,$$

leading to

$$\sigma_3^c = 0$$

which is a contradiction.

□

Lemma 12.2.41. *Let*

$$\varphi : M_{1,c}(\mathtt{stand})[-p] \to M_{1,c}(\mathtt{stand})/\langle v_{p-3c}, v_{p+3c}\rangle$$

be the map of $H_{1,c}(S_3,\mathfrak{h})$ modules given by $\varphi(f\otimes b_\pm) = f\cdot v_\pm$. Then $\varphi(v_{p-3c}) \neq 0$.

Proof. Let $\psi : M_{1,c}(\mathtt{stand})[-p] \to M_{1,c}(\mathtt{stand})$ be the map of $H_{1,c}(S_3,\mathfrak{h})$ modules given by the same formula, so that $\varphi(v_{p-3c}) = 0$ if and only if

$$\varphi(v_{p-3c}) = A\cdot v_{p-3c} + B\cdot v_{p+3c}$$

for some $A, B \in S(\mathfrak{h}^*)$. Considering their behaviour under S_3, we see that $A \in S(\mathfrak{h}^*)^{S_3}$ and $B = B'q$ for $B' \in S(\mathfrak{h}^*)^{S_3}$. We then calculate modulo σ_2, using the explicit expressions for $v_\pm, v_{p-3c}, v_{p+3c}$ from Lemma 12.2.6, and distinguishing two cases.

1. If $p \equiv 1 \pmod 3$, then

$$v_+ = \sigma_3^{\frac{p-1}{3}}(-b_+\otimes b_+ + 3b_-\otimes b_-) \bmod \sigma_2$$
$$v_- = \sigma_3^{\frac{p-1}{3}}(b_+\otimes b_- + b_-\otimes b_+) \bmod \sigma_2$$
$$v_{p-3c} = \sigma_3^{\frac{p-1}{3}-c}(b_+\otimes b_+ + 3b_-\otimes b_-) \bmod \sigma_2.$$

   We calculate modulo σ_2:

$$\varphi(v_{p-3c}) = \sigma_3^{\frac{p-1}{3}-c}(b_+\cdot v_+ + 3b_-\cdot v_-) \bmod \sigma_2 \\ = \sigma_3^{2\frac{p-1}{3}-c}((-b_+^2+3b_-^2)\otimes b_+ + 3(2b_+b_-)\otimes b_-) \neq 0 \bmod \sigma_2,$$

   which proves the claim.

2. If $p \equiv 2 \pmod 3$, then

$$v_+ = \sigma_3^{\frac{p-2}{3}}(-(-b_+^2+3b_-^2)\otimes b_+ + 3(2b_+b_-)\otimes b_-) \bmod \sigma_2$$

$$v_- = \sigma_3^{\frac{p-2}{3}}((-b_+^2+3b_-^2)\otimes b_- + (2b_+b_-)\otimes b_+) \bmod \sigma_2$$
$$v_{p-3c} = \sigma_3^{\frac{p-2}{3}-c}((-b_+^2+3b_-^2)\otimes b_+ + 3(2b_+b_-)\otimes b_-) \bmod \sigma_2.$$

Similarly,

$$\begin{aligned}\varphi(v_{p-3c}) &= \sigma_3^{\frac{p-2}{3}-c}((-b_+^2+3b_-^2)\cdot v_+ + 3(2b_+b_-)\cdot v_-) \bmod \sigma_2\\ &= \sigma_3^{2\frac{p-2}{3}-c}\Big((-(-b_+^2+3b_-^2)^2+3(2b_+b_-)^2)\otimes b_+\\ &\qquad + 6(2b_+b_-)(-b_+^2+3b_-^2)\otimes b_-\Big) \bmod \sigma_2\\ &= \sigma_3^{2\frac{p-2}{3}-c}(2\cdot 108\sigma_3 b_+\otimes b_+ + 6\cdot 108\sigma_3 b_-\otimes b_-) \bmod \sigma_2\\ &= 216\sigma_3^{2\frac{p-2}{3}-c+1}(b_+\otimes b_+ + 3b_-\otimes b_-)\neq 0 \bmod \sigma_2.\end{aligned}$$

□

Lemma 12.2.42. *The coefficient of z^{2p-1} in the power series expansion of*

$$\frac{1}{(1-z)^2}\left(2-z^{p-3c}-2\cdot z^p - z^{p+3c}\right)$$

around $z=0$ is 0.

Proof. The coefficient of z^k in $\frac{1}{(1-z)^2}$ is $k+1$, so the coefficient of z^{2p-1} in the above expression is

$$2\cdot(2p-1+1)-(2p-1-(p-3c)+1)-2\cdot(2p-1-p+1)-(2p-1-(p+3c)+1)=0.$$

□

At this point in the proof of Theorem 12.2.2 we conclude that the Hilbert polynomial of the irreducible module $L_{1,c}(\texttt{stand})$ has the form (12.2.4). The following Lemma shows that the only module with such Hilbert polynomial is $M_{1,c}(\texttt{stand})/\langle v_{p-3c}, v_+, v_-, v_{p+3c}\rangle$ itself.

Lemma 12.2.43. *Let $p>3$ and suppose that $0<c<p/3$. Suppose that Q is an $H_{1,c}(S_3,\mathfrak{h})$ module which is a quotient of $M_{1,c}(\texttt{stand})/\langle v_{p-3c}, v_+, v_-, v_{p+3c}\rangle$, with Hilbert polynomial equal to*

$$\begin{aligned}Hilb_Q(z) &= Hilb_{M_{1,c}(\texttt{stand})/\langle v_{p-3c},v_+,v_-,v_{p+3c}\rangle}(z) - M\cdot z^p\cdot Hilb_{L_{1,c}(\texttt{stand})}(z)-\\ &\quad - K\cdot z^{3c+p}\cdot Hilb_{L_{1,c}(\texttt{sign})}(z) - N\cdot z^{2p-3c}Hilb_{L_{1,c}(\texttt{triv})}(z)+O(z^{2p})\end{aligned}$$

for some $M,N,K\geq 0$. Then:

1. *$M=0$;*

2. $K = N = 0$;

3. $Q = M_{1,c}(\texttt{stand})/\langle v_{p-3c}, v_+, v_-, v_{p+3c}\rangle$.

Proof. 1. We distinguish two cases.

(a) If $0 < c < p/6$, we calculate the dimension of Q^{2p-3c}. We have $p + 3c < 2p - 3c$, so

$$\begin{aligned}\dim Q^{2p-3c} &= \dim M_{1,c}^{2p-3c}(\texttt{stand})/\langle v_{p-3c}, v_+, v_-, v_{p+3c}\rangle \\ &\quad - M \cdot \dim L_{1,c}^{p-3c}(\texttt{stand}) - K \cdot \dim L_{1,c}^{p-6c}(\texttt{sign}) - N \cdot \dim L_{1,c}^{0}(\texttt{triv}).\end{aligned}$$

By equation (12.2.3) we have that $\dim(M_{1,c}^{2p-3c}(\texttt{stand})/\langle v_{p-3c}, v_+, v_-, v_{p+3c}\rangle)$ equals

$$2\cdot(2p-3c+1)-(2p-3c-(p-3c)+1)-2\cdot(2p-3c-p+1)-(2p-3c-(p-3c)+1) = 6c-2.$$

Equation (12.2.4) lets us conclude that

$$\begin{aligned}\dim L_{1,c}^{p-3c}(\texttt{stand}) &= \dim M_{1,c}^{p-3c}(\texttt{stand})/\langle v_{p-3c}, v_+, v_-, v_{p+3c}\rangle \\ &= 2(p-3c+1) - 1 = 2p - 6c + 1.\end{aligned}$$

By Theorem 12.0.1 $\mathrm{Hilb}_{L_{1,c}(\texttt{sign})} = \left(\frac{1-z^{p-3c}}{1-z}\right)$ so

$$\dim L_{1,c}^{p-6c}(\texttt{sign}) = (p - 6c + 1) - 2 = p - 6c - 1.$$

We also know that $\dim L_{1,c}^{0}(\texttt{triv}) = 1$. Putting it all together we get:

$$0 \leq \dim Q^{2p-3c} = (p-1) - M \cdot (2p - 6c + 1) - K \cdot (p - 6c - 1) - N \cdot 1.$$

Rearranging and using $c < p/6$ we get

$$\begin{aligned}p - 2 &> 6c - 2 \\ &\geq M \cdot (2p - 6c + 1) + K \cdot (p - 6c + 1) + N \\ &\geq M \cdot (2p - 6c + 1) \\ &> M \cdot (p + 1).\end{aligned}$$

The only nonnegative integer M satisfying this is $M = 0$.

(b) If If $p/6 < c < p/3$, we have $2p - 3c < p + 3c$ and similarly calculate the dimension of Q^{p+3c}, getting:

$$\begin{aligned}0 \leq \dim Q^{p+3c} &= \dim M_{1,c}^{p+3c}(\texttt{stand})/\langle v_{p-3c}, v_+, v_-, v_{p+3c}\rangle \\ &\quad - M \cdot \dim L_{1,c}^{3c}(\texttt{stand}) - K \cdot \dim L_{1,c}^{0}(\texttt{sign}) - N \cdot \dim L_{1,c}^{6c-p}(\texttt{triv})\end{aligned}$$

$$\begin{aligned}
&= (2 \cdot (p + 3c + 1) - (p + 3c - p + 3c + 1) - 2(p + 3c - p + 1) - 1) \\
&\quad - M \cdot (2 \cdot (3c + 1) - (3c - p + 3c + 1)) - K \cdot 1 - N \cdot (6c - p + 1) \\
&= (2p - 6c - 2) - M \cdot (p + 1)) - K - N \cdot (6c - p + 1) \\
&\leq (2p - 6c - 2) - M \cdot (p + 1) \,.
\end{aligned}$$

We rearrange this as

$$M \cdot (p + 1) \leq 2p - 6c - 2 < p + 6c - 6c - 2 = p - 2,$$

and notice that the only nonnegative integer M satisfying this is $M = 0$.

2. Using part (1), we calculate the dimension in the degree $2p - 1$ graded piece of Q as

$$\begin{aligned}
\dim Q^{2p-1} &= \dim M_{1,c}^{2p-1}(\texttt{stand}) / \langle v_{p-3c}, v_+, v_-, v_{p+3c} \rangle - \\
&\quad - K \cdot \dim L_{1,c}^{p-3c-1}(\texttt{sign}) - N \cdot \dim L_{1,c}^{3c-1}(\texttt{triv}) \\
&= (2 \cdot 2p - (2p - 1 - p + 3c + 1) - 2 \cdot (2p - 1 - p + 1) - (2p - 1 - p - 3c + 1)) - \\
&\quad - K \cdot (p - 3c - 1 + 1) - N \cdot (3c - 1 + 1) \\
&= -K \cdot (p - 3c) - N \cdot 3c.
\end{aligned}$$

Using that $\dim Q^{2p-1} \geq 0$, $p - 3c > 0$ and $3c > 0$, the only nonnegative K, N satisfying this are $K = N = 0$.

3. Follows from parts (1) and (2).

□

We end this section with a conjecture.

Conjecture 12.2.44. *The irreducible representation $L_{1,c}(\texttt{stand})$ of the rational Cherednik algebra $H_{1,c}(S_3, \mathfrak{h})$ over an algebraically closed field of characteristic $p > 3$ for $c \in \mathbb{F}_p$, $p/3 < c < p/2$, is the quotient of the Verma module $M_{1,c}(\texttt{stand})$ by the submodule generated by vectors in degrees:*

- $-p + 3c$ *- one dimensional space, of type* `sign`
- $3p - 3c$ *- one dimensional space, of type* `triv`
- $p + 3c$ *- one dimensional space, of type* `sign`
- $5p - 3c$ *- one dimensional space, of type* `triv`

Its character and Hilbert polynomial are

$$\chi_{L_{1,c}(\texttt{stand})}(z) = \chi_{S\mathfrak{h}^*}(z) \cdot \Big([\texttt{stand}] - [\texttt{sign}] z^{-p+3c} - [\texttt{triv}] z^{3p-3c}$$

$$- [\mathtt{sign}]z^{p+3c} - [\mathtt{triv}]z^{5p-3c} + [\mathtt{stand}]z^{4p}\Big)$$

$$Hilb_{L_{1,c}(\mathtt{stand})}(z) = \frac{2 - z^{-p+3c} - z^{3p-3c} - z^{p+3c} - z^{5p-3c} + 2z^{4p}}{(1-z)^2}.$$

Notice that the vectors v_+, v_- which are singular in the generic case are contained in the submodule generated in degrees $-p+3c$ and $3p-3c$. We believe that this conjecture can be proven by the same methods used throughout this chapter, however the proof is likely to be lengthy. Supporting evidence for this conjecture comes from the computer code in Appendix A.2. The output of that code tells us in which degrees singular vectors occur, for fixed p and c. The conjectured formula for the degrees fits a pattern in the data. The Casimir operator Ω tells us how the singular vectors in a given degree behave as S_3-representations and allows us to conjecture the character of the quotient.

Appendix A

Magma code

One approach to finding singular vectors is the consider a particular bilinear form, the kernel of which is the maximal proper graded submodule $J_{t,c}(\tau)$ of a Verma module. This bilinear form, denoted B, is described in Section 2.5 [BaCh13a]. The form is graded so its kernel in each degree may be considered, and the generators of these kernels are singular vectors. Using the symbolic programming language [Magma], we calculated the Gram matrices of the form B and their kernels. We did this for many choices of parameters to generate data, from which we were able to identify patterns and learn information about the singular vectors.

A.1 Magma code for $M_{t,c}(S_3, \texttt{triv})$

The following code produces Gram matrices of the bilinear form B.

```
p := 7;                          // field characteristic
k := AlgebraicClosure(GF(p)); // algebraic closure of the finite field F_p
K<c> := FunctionField(k);     // let K = k(c)

n := 3;        // size of symmetric group
G := Sym(n);   // S_n

t := 1;

// We form the natural permutation module V, and use it define the action on
    our modules
V := PermutationModule(G,K);
M<[x]>:= PolynomialRing(K,n);// PolynomialRing in n variables x[1], ..., x[n]
A<[y]>:= PolynomialRing(K,n);// PolynomialRing in n variables y[1], ..., y[n]


// Auxiliary ring, used to produce the monomial bases of S(h*) / S(h)
P := PolynomialRing(K,n-1);
xdrag := hom< P -> M | x[1],x[2] >;
ydrag := hom< P -> A | y[1]-y[3], y[2]-y[3] >;
```

```


// This is the quotient map S(V*) ->> S(h*). Here it is defined on
// S(V*) = K[ x[1], x[2], x[3] ]. It is identity on x[1], x[2],
// and sends x[3] to -x[1]-x[2]
quot := hom< M -> M | x[1], x[2], -x[1]-x[2]>;



ActOnTerm := function(s, term)
// input
//  - s: an element of G
//  - term: a single term of M = K[ x[1], x[2], x[3] ]
// output
//  returns the value of s acting on term, using the permutation action of S_3 on M

  // we define the group action recursively, based on the degree of the term
  case TotalDegree(M!term):
    when 0:
      // group acts trivially on constants
      return term;

    when 1:
      // iterate i between 1 and n to determine the monomial
      for i in [1 .. n] do
        case LeadingMonomial(term):
          when x[i]:
          // in this case, our term is of the form coefficient*x[i],
          // use the permutation module V to complete the action
            return LeadingCoefficient(term)*Polynomial(ElementToSequence((V.i)*s),SetToSequence(MonomialsOfDegree(M,1)));
        end case;
      end for;

    else:
      // for higher order terms, use recursion
      for i in [1 .. n] do
        divisible,quotient := IsDivisibleBy(term,x[i]);
        if divisible then
          return $$(s,x[i])*$$(s,quotient);
          break;
        end if;
      end for;
  end case;
end function;


```

```
Act := function(s,f)
// input
//   - s: an element of G
//   - f: a polynomial in M = K[ x[1], x[2], x[3] ]
// output
//   returns the value of s acting on f, using the permutation action of S_3

  sf := M!0;
  for eachterm in Terms(f) do
    // form the result by acting on each term of f, using the ActOnTerm
      function
    sf := sf + ActOnTerm(s,eachterm);
  end for;
  return sf;
end function;






PairTerms := function(ty,tx)
// input
//   - ty: a single linear (degree 1) term of A = K[ y[1], y[2], y[3] ]
//   - tx: a single linear (degree 1) term of M = K[ x[1], x[2], x[3] ]
// output
//   returns the natural pairing of <ty, tx>

  a := -1;
  b := -1;
  for i,j in [1 .. n] do
    if LeadingMonomial(ty) eq y[i] then a := i; end if;
    if LeadingMonomial(tx) eq x[j] then b := j; end if;
  end for;
  assert not a eq -1;
  assert not b eq -1;

  return (a eq b) select LeadingCoefficient(ty)*LeadingCoefficient(tx) else
    0;
end function;
```

```
Pair := function(fy, fx)
// input
//   - fy: any linear element of A = K[ y[1], y[2], y[3] ]
//   - fx: any linear element of M = K[ x[1], x[2], x[3] ]
// output
//   returns the natural pairing <fy, fx> by considering them as elments of h/
    h*

  result := K!0;
  // by bilinearity, obtain result by considering each pair of terms
  for yterm in Terms(fy) do
  for xterm in Terms(fx) do
    result := result + PairTerms(yterm,xterm);
  end for;
  end for;
  return result;
end function;


d := function(y,term)
// input
//   - y: a linear term of A = K[ y[1], y[2], y[3] ]
//   - term: a single term of M = K[ x[1], x[2], x[3] ]
// output
//   returns d_y(term), the partial derivative of the term with respect to y

  case TotalDegree(term):
    when 0:
      // derivative of constant is zero
      return 0;

    when 1:
      // derivative of linear term is given by pairing it with y
      return Pair(y,term);

    else:
      // for higher order terms, use Leibniz rule for derivatives
      for i in [1 .. n] do
        divisible, quotient := IsDivisibleBy(term,x[i]);
        if divisible then
          return ( Pair(y,x[i])*quotient + x[i]*$$(y,quotient) );
        end if;
      end for;
  end case;
end function;
```

```
Deriv := function(y, f)
// input
//   - y: a linear term of A = K[ y[1], y[2], y[3] ]
//   - f: a polynomial in M = K[ x[1], x[2], x[3] ]
// output
//   returns d_y(f), the partial derivative of f with respect to y

  result := M!0;
  // calculate the result by differentiating each term of f using the
     function d
  for eachterm in Terms(f) do
    result := result + d(y,eachterm);
  end for;
  return result;
end function;



Dunkl := function(y,f)
// input
//   - y: a linear term of A = K[ y[1], y[2], y[3] ]
//   - f: a polynomial in M = K[ x[1], x[2], x[3] ]
// output
//   returns the value of D_y(f), the Dunkl operator D_y applied to f

  // The Dunkl operator has two parts, a derivative an a sum.
  // We calculate these separately.
  deriv := Deriv(y,f);

  summands := [**];
  // We must sum over reflections (ij) in G where 1 \leq i < j \leq n
  for i,j in [1 .. n] do
    if i lt j then
      s := G!(i,j); // the transposition s=(ij) is a reflection in S_n
      alpha := x[i] - x[j];
      summand := Pair(y,alpha)*(f - Act(s,f)) div (alpha);
    Append(~summands,summand);
    end if;
  end for;

  sum := M!0;
  for each in summands do sum := sum + each; end for;
  return t*deriv - c*sum;
end function;
```

```
// With all the previous functions implemented, we are ready to implement the
     bilinear form B, in order to calculate the singular vectors.



Bt := function(ty,tx)
// input
//   - ty: a single term of A = K[ y[1], y[2], y[3] ]
//   - tx: a single term of M = K[ x[1], x[2], x[3] ]
// output
//   returns the value of B(ty, tx), using recursion.

  // B should return zero if either input is zero
  if (ty eq A!0) or (tx eq M!0) then return 0; end if;

  // B should only be calculated on inputs of the same degree
  assert TotalDegree(A!ty) eq TotalDegree(M!tx);

  // if both terms are in degree 0, return their product, since B(1,1)=1
  if (TotalDegree(A!ty) eq 0) and (TotalDegree(M!ty) eq 0) then
    return K!tx*(M!ty);

  else
  // for higher order terms, use the recursive definition of B, by Dunkl
    operators
    for i in [1 .. n] do
      divisible, quotient := IsDivisibleBy(ty,y[i]);
      if divisible then
      // WARNING: this function only accepts single terms as input
      // the result of y[i].tx may contain more than one term
      // carefully compute the form on each term of y[i].tx using recursion
        yf := Dunkl(y[i],tx);
        answer := K!0;
          for term in Terms(yf) do
            answer := answer + $$(quotient, term);
          end for;
        return answer;
        end if;
    end for;
  end if;
end function;
```

```
B := function(fy,fx)
// input
//   - fy: any element of A = K[ y[1], y[2], y[3] ]
//   - fx: any element of M = K[ x[1], x[2], x[3] ]
// output
//   returns the value of B(fy, fx).

  answer := K!0;
    for yterm in Terms(fy) do
    for xterm in Terms(fx) do
      answer := answer + Bt(yterm, xterm);
    end for;
    end for;
  return answer;
end function;


// Using our bilinear form B, we now calculate the Gram matrices, by using
    the standard monomial bases in each degree of S(h) and S(h*)
// In degree 0, the matrix is identity, on account of the dual bases chosen
B0 := ScalarMatrix(dimtau, K!1);

// for storing the Gram matrix of the form B in each degree
matrices := [**];
// for storing the monomial bases in each degree of K[ x[1], x[2] ]
xbases := [**];
// for the monomial bases in each degree of  K[(y[1]-y[3]), (y[2] - y[3])]
ybases := [**];

for degree in [1] do
    // compute the monomials of degree 1 in 2 variables, using P
  monomials := MonomialsOfDegree(P,degree);
  // using ydrag, map these monomials into monomials of
  // K[(y[1]-y[3]), (y[2] - y[3])] and store in ybases
  ybases[degree] := ydrag(monomials);
  // using ydrag, map these monomials into monomials of
  // K[ x[1], x[2] ] and store in xbases
  xbases[degree] := xdrag(monomials);

  matrixEntries := [];
  // to calculate the entries of the matrix, iteratate over the bases and
    pair them using B
    for j in [1 .. #xbases[degree]] do
    for i in [1 .. #ybases[degree]] do
      Append(~matrixEntries, <i,j,B(xbases[degree][j],ybases[degree][i])>);
    end for;
    end for;
```

```
  // compiles the matrixEntries we calculated into a matrix
  matrix := Matrix(#ybases[degree],#xbases[degree],matrixEntries);
  // save the matrix to the matrices array.
  matrices[degree] := matrix;

  // save the matrix to a file "B1"
  SetOutputFile("B"*Sprint(degree));
  print matrix;
  UnsetOutputFile();
end for;

for degree in [2 .. 5*p] do
  // compute the monomials of current degree in 2 variables, using P
  monomials := MonomialsOfDegree(P,degree);
  // using ydrag, map these monomials into monomials of
  // K[(y[1]-y[3]), (y[2] - y[3])] and store in ybases
  ybases[degree] := ydrag(monomials);
  // using ydrag, map these monomials into monomials of
  // K[ x[1], x[2] ] and store in xbases
  xbases[degree] := xdrag(monomials);

  matrixEntries := [];
  // calculate matrix entries, using values from the previously computed
   matrix
    for i in [1 .. #ybases[degree]] do
    for j in [1 .. #xbases[degree]] do
      for ymonomial in ydrag(MonomialsOfDegree(P,1)) do
        divisible,quotient := IsDivisibleBy(ybases[degree][i], ymonomial);
        if divisible then
          row := Position(ybases[degree-1],quotient);
          yx := quot(Dunkl(ymonomial,xbases[degree][j]));
          if yx eq M!0 then
            entry := K!0;
          else
            col := Transpose(Matrix(K,vector(yx)));
            entry := (((b[degree-1])[row])*col)[1];
          end if;

          Append(~matrixEntries,<i,j,entry>);
          break;
        end if;
      end for;
    end for;
    end for;
  // compiles the matrixEntries we calculated into a matrix
  matrix := Matrix(#ybases[degree],#xbases[degree],matrixEntries);
  b[degree] := matrix;

```

```
  // save the matrix to the matrices array.
  matrices[degree] := matrix;

  // save the matrix to a file "Bd" where d is the current degree
  SetOutputFile("B"*Sprint(degree));
  print mat;
  UnsetOutputFile();
end for;
```

A.2 Magma code for $M_{t,c}(S_3, \text{stand})$

```
p := 11;                          // field characteristic
k := AlgebraicClosure(GF(p)); // algebraic closure of the finite field F_p
K<c> := FunctionField(k);      // let K = k(c)

n := 3;        // size of symmetric group
G := Sym(n);   // S_n

t := 1;


// We form the natural permutation module V, and use it define the action on
   our modules
V := PermutationModule(G,K);


// PolynomialRing which represents the structure S(V*) \otimes V. The
// notation [1,1,1,0,0,0] sets the "weights" of the variables, with x[1],
// x[2], x[3] having weight 1, and other variables have weight 0. This allows
// us to define the degree of a vector appropriately.
M<[x]> := PolynomialRing(K,[1,1,1,0,0,0]);
AssignNames(~M,["x[1]","x[2]","x[3]","o(x1)","o(x2)", "o(x3)"]);



// PolynomialRing which represent S(V) \otimes h. We then get the ring
// S(h) \otimes h by considering the subring spanned by y[1]-y[3], y[2]-y[3]
// in the first tensor leg.
A<[y]> := PolynomialRing(K,[1,1,1,0,0]);
AssignNames(~A,["y[1]","y[2]","y[3]","o(y1-y3)","o(y2-y3)"]);


```

```
// Auxiliary ring, used to produce the monomial bases of S(h*) / S(h)
P := PolynomialRing(K,n-1);
xdrag := hom< P -> M | x[1],x[2] >;
ydrag := hom< P -> A | y[1]-y[3], y[2]-y[3] >;


// This is the quotient map S(V*) \otimes V* ->> S(h*) \otimes \tau. On both tensor legs, it is identity on x[1], x[2], and sends x[3] to -x[1]-x[2].
quot := hom< M -> M | x[1], x[2], -x[1]-x[2], x[4], x[5], -x[4] - x[5]>;

// This is the quotient map S(V*) \otimes V* ->> S(V*) \otimes \tau. It is identity on the first tensor leg, and on the second tensor leg it is identity on x[1], x[2], and sends x[3] to -x[1]-x[2].
standquot := hom< M -> M | x[1], x[2], x[3], x[4], x[5], -x[4] - x[5]>;

// This is the quotient map S(V*) \otimes V* ->> S(h*) \otimes V*. On the first tensor leg, it is identity on x[1], x[2], and sends x[3] to -x[1]-x[2]. On the second tensor leg it is identity.
polyquot := hom< M -> M | x[1], x[2], -x[1]-x[2], x[4], x[5], x[6]>;



ActOnTerm := function(s, term)
// input
//   - s: an element of G
//   - term: a single term of K[ x[1], x[2], x[3] ]
// output
//   returns the value of s acting on term, using the permutation action of S_3 on M
// The variables x[4], x[5], x[6] are not expected to appear in the input.

  // we define the group action recursively, based on the degree of the term
  case WeightedDegree(M!term):
    when 0:
      // group acts trivially on constants
      return term;

    when 1:
      // iterate i between 1 and n to determine the monomial
      for i in [1 .. n] do
        case LeadingMonomial(term):
          when x[i]:
            // in this case, our term is of the form coefficient*x[i],
            // use the permutation module V to complete the action
            return LeadingCoefficient(term)*Polynomial(ElementToSequence((V.i)*s),SetToSequence(MonomialsOfWeightedDegree(M,1)));
        end case;
      end for;
```

```
    else:
      // for higher order terms, use recursion
      for i in [1 .. n] do    // iterate i between 1 and n
      divisible,quotient := IsDivisibleBy(term,x[i]); // check if term is
    divisible by x[i]
        if divisible then // if so,
          return $$(s,x[i])*$$(s,quotient); // action is group like, s.(x[1]x
    [2])=(s.x[1])(s.x[2])
          break;
        end if;
      end for;
  end case;
end function;


PolynomialAct := function(s,f)
// input
//  - s: an element of G
//  - f: a polynomial in M = K[ x[1], x[2], x[3] ]
// output
//  returns the value of s acting on f, using the permutation action of S_3
// The variables x[4], x[5], x[6] are not expected to appear in the input.

  sf := M!0;
  for eachterm in Terms(f) do
    // form the result by acting on each term of f, using the ActOnTerm
    function
    sf := sf + ActOnTerm(s,eachterm);
  end for;
  return sf;
end function;


ActOnV := function(s,v)
// input
//  - s: an element of G
//  - v: a linear term in V, considered as an element in K[ x[4], x[5], x[6]
    ] of the form a*x[4] + b*x[5] + c*x[6]
// output
//  returns the value of s acting on v, using the permutation action of S_3

  sv := ( V!([ Coefficient(v,x[4],1), Coefficient(v,x[5],1), Coefficient(v,x
    [6],1) ]) )* s;
  return Polynomial( ElementToSequence(sv) , [x[4], x[5], x[6]] );
end function;



```

```
ActOnVector := function(s, fv)
// input
//  - s: an element of G
//  - fv: an element of M = K[x[1], x[2], x[3], x[4], x[5], x[6] ]
// output
//  returns the value of s acting on the vector fv

  // complete the action by considering the \otimes x[1], \otimes x[2], and \
    otimes x[3] components of fv
  f1:=Coefficient(fv,x[4],1); // component attached to \otimes x[1]
  f2:=Coefficient(fv,x[5],1); // component attached to \otimes x[2]
  f3:=Coefficient(fv,x[6],1); // component attached to \otimes x[3]

  return ActOnPolynomial(s,f1)*ActOnV(s,x[4]) + ActOnPolynomial(s,f2)*ActOnV(
    s,x[5]) + ActOnPolynomial(s,f3)*ActOnV(s,x[6]);
end function;




PairTerms := function(ty,tx)
// input
//  - ty: a single linear (degree 1) term of A = K[ y[1], y[2], y[3] ]
//  - tx: a single linear (degree 1) term of M = K[ x[1], x[2], x[3] ]
// output
//  returns the natural pairing of <ty, tx> by considering them as elements
    of V/V*

  a := -1;
  b := -1;
  for i,j in [1 .. n] do
    if LeadingMonomial(ty) eq y[i] then a := i; end if;
    if LeadingMonomial(tx) eq x[j] then b := j; end if;
  end for;
  assert not a eq -1;
  assert not b eq -1;

  return (a eq b) select LeadingCoefficient(ty)*LeadingCoefficient(tx) else
    0;
end function;
```

```
Pair := function(fy, fx)
// input
//   - fy: any linear element of A = K[ y[1], y[2], y[3] ]
//   - fx: any linear element of M = K[ x[1], x[2], x[3] ]
// output
//   returns the natural pairing <fy, fx> by considering them as elments of h/
    h*

  result := K!0;
  // by bilinearity, obtain result by considering each pair of terms
  for yterm in Terms(fy) do
  for xterm in Terms(fx) do
    result := result + PairTerms(yterm,xterm);
  end for;
  end for;
  return result;
end function;


d := function(y,term)
// input
//   - y: an element of h, interpreted as a linear term of A = K[ y[1], y[2],
    y[3] ]
//   - term: a single term of M = K[ x[1], x[2], x[3] ]
// output
//   returns d_y(term), the partial derivative of the term with respect to y

  case WeightedDegree(term):
    when 0:
      // derivative of constant is zero
      return 0;

    when 1:
      // derivative of linear term is given by pairing it with y
      return Pair(y,term);

    else:
      // for higher order terms, use Leibniz rule for derivatives
      for i in [1 .. n] do
        divisible, quotient := IsDivisibleBy(term,x[i]);
        if divisible then
          return ( pair(y,x[i])*quotient + x[i]*$$(y,quotient) );
        end if;
      end for;
  end case;
end function;
```

```

Deriv := function(y, fx)
// input
//   - y: an element of V, interpreted as a linear term of A = K[ y[1], y[2], y[3] ]
//   - f: a polynomial in M = K[ x[1], x[2], x[3] ]
// output
//   returns d_y(f), the partial derivative of f with respect to y
// The variables x[4], x[5], x[6] are not expected to appear in the input.
  result := M!0;
  for term in Terms(fx) do
    result := result + d(y,term);
  end for;
  return result;
end function;



Dunkl := function(y,fv)
// input
//   - y: a linear term of K[ y[1], y[2], y[3] ], understood as an element of V
//   - fv: a vector in M = K[ x[1], x[2], x[3], x[4], x[5], x[6] ], understood as an element of SV* \otimes V*
// output
//   returns the value of D_y(f), the Dunkl operator D_y applied to f

  // The input vector is of the form fv = f1 \otimes x[1] + f2 \otimes x[2] + f3 \otimes x[3]
  f1:=Coefficient(fv,x[4],1);
  f2:=Coefficient(fv,x[5],1);
  f3:=Coefficient(fv,x[6],1);

  df := Deriv(y,f1)*x[4] + Deriv(y,f2)*x[5] + Deriv(y,f3)*x[6];

  summands := [**];
  // We sum over reflections s=(ij) with 1 \leq i < j \leq n

  // We compute the Dunkl operator on the \otimes x[1] component
  for i,j in [1 .. n] do
    if i lt j then // we have a transposition s=(ij) and \alpha_s = x_i - x_j
      s := G!(i,j);
      alpha := x[i] - x[j];
      summand := ( Pair(y,alpha)*(f1 - ActOnPolynomial(s,f1)) div (alpha) )* ActOnV(s,x[4]);
      Append(~summands,summand);
    end if;
  end for;
```

```
  // We compute the Dunkl operator on the \otimes x[2] component
  for i,j in [1 .. n] do
    if i lt j then
      s := G!(i,j);
      alpha := x[i] - x[j];
      summand := ( Pair(y,alpha)*(f2 - ActOnPolynomial(s,f2)) div (alpha) )*
    ActOnV(s,x[5]);
      Append(~summands,summand);
    end if;
  end for;

  // We compute the Dunkl operator on the \otimes x[3] component
  for i,j in [1 .. n] do
    if i lt j then
      s := G!(i,j);
      alpha := x[i] - x[j];
      summand := ( Pair(y,alpha)*(f3 - ActOnPolynomial(s,f3)) div (alpha) )*
    ActOnV(s,x[6]);
      Append(~summands,summand);
    end if;
  end for;

  sum := M!0;
  for summand in summands do sum := sum + summand; end for;

  return t*df - c*sum;
end function;






Bt := function(tx,ty)
// input
//  - ty: a single term of A = K[ y[1], y[2], y[3], y[4], y[5] ]
//  - tx: a single term of M = K[ x[1], x[2], x[3], x[4], x[5], x[6] ]
// output
//  returns the value of B(ty, tx), using recursion.

  // B should return zero if either input is zero
  if (ty eq A!0) or (tx eq M!0) then return 0; end if;

  // B should only be calculated on inputs of the same degree
  assert WeightedDegree(A!ty) eq WeightedDegree(M!tx);

```

```
  // In degree 0, B is given by the pairing of tau and tau*.
  if (WeightedDegree(A!ty) eq 0) and (WeightedDegree(M!tx) eq 0) then
    for i in [4, 5, 6] do
      case LeadingMonomial(tx):
        when x[i]:
      case LeadingMonomial(ty):
        when y[4]:
          return LeadingCoefficient(tx)*LeadingCoefficient(ty)*pair(y[1]-y
    [3],x[i-3]);
        when y[5]:
          return LeadingCoefficient(tx)*LeadingCoefficient(ty)*pair(y[2]-y
    [3],x[i-3]);
      end case;
      end case;
    end for;

  else
    for i in [1,2,3] do
      divisible, quotient := IsDivisibleBy(ty,y[i]);
      // WARNING: this function only accepts single terms as input
      // the result of y[i].tx may contain more than one term, so
      // recursively compute the form on each term of y[i].tx
      if divisible then
        ytx := Dunkl(y[i],tx);
        answer := K!0;
        for term in Terms(ytx) do
          answer := answer + $$(term, quotient);
        end for;
        return answer;
      end if;
    end for;
  end if;
end function;


B := function(fx, fy)
  answer := K!0;
  for yterm in Terms(fy) do
  for xterm in Terms(fx) do
    answer := answer + Bt(xterm, yterm);
  end for;
  end for;
  return answer;
end function;
```

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